Deep Inference Proof Systems for Monotone Quantifiers

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- Rules for Quantifiers ?
- Monotonicity and Deep Inference

2 System QC
- Rules and Derivations
- Properties

3 Extensions and Applications
- Extending QC
- n-ary Quantifiers
- Notable Quantifier Classes

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Well-known extension of usual quantification: quantifier $\approx$ relation on relations.

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A quantifier of type $\langle k_1, ..., k_n\rangle$ takes n formulae as arguments, and binds k_n variables in each formula:

$$\mathcal{Q}\overrightarrow{x_1}, ..., \overrightarrow{x_n}[\varphi_1(\overrightarrow{x_1}), ..., \varphi_n(\overrightarrow{x_n})]$$

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Interpreted in a model $\mathcal{M}$ by $[\![\mathcal{Q}]\!]^{\mathcal{M}} \subseteq \mathcal{P}(\mathcal{M}^{k_1}) \times ... \times \mathcal{P}(\mathcal{M}^{k_n})$:

$$\mathcal{M} \models \mathcal{Q}\overrightarrow{x_1}, ..., \overrightarrow{x_n}[\varphi_1, ..., \varphi_n]$$

iff

$$([\![\varphi_1]\!]^{\mathcal{M}}, ..., [\![\varphi_n]\!]^{\mathcal{M}}) \in [\![\mathcal{Q}]\!]^{\mathcal{M}}$$

Generalized Quantifiers

Examples:

- $\exists$ is the type $\langle 1 \rangle$ quantifier defined by the unary relation "being non-empty":

$$[\![\exists]\!]^{\mathcal{M}} = \{X \in \mathcal{P}(\mathcal{M}) \mid X \neq \emptyset\}$$

- "Most X are Y" is the type $\langle 1, 1 \rangle$ quantifier:

$$[\![Most]\!]^{\mathcal{M}} = \{(X, Y) \in \mathcal{P}(\mathcal{M}) \times \mathcal{P}(\mathcal{M}) \mid |A \cap B| > |A - B|\}$$

- Henkin quantifiers, e.g. with type $\langle 4 \rangle$:

$$[\![\left(\begin{smallmatrix} \forall x & \exists y \\ \forall u & \exists v \end{smallmatrix}\right)]\!]^{\mathcal{M}} = \{R \in \mathcal{P}(\mathcal{M}^4) \mid \exists f \exists g \forall x \forall u, (x, y, f(x), g(y)) \in R\}$$

- Many, many more...

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Very good notions of semantics, very poor notions of proofs *in general*.

Proof Rules for Quantifiers

The usual quantifiers have very convenient introduction/elimination rules e.g.

$$\exists \frac{A(t)}{\exists x A(x)} \qquad\qquad \forall \frac{A(x)}{\forall x A(x)} \text{, with } x \text{ eigenvariable}$$

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Generalized quantifiers have no such rules in general.
Example (Keisler) : "uncountably many" $Q^{>\omega}$:

$$\times\,\frac{A(t)}{Q^{>\omega}x[A(x)]} \qquad \times\,\frac{Q^{>\omega}x[A(x)]}{A(t)}$$

are not valid rules.

What can we do ?

Rules for Quantifiers ?

A valid pseudo-syllogism:

$$\begin{array}{c} Most\ C\ are\ A \\ All\ A\ are\ B \\ \therefore Most\ C\ are\ B \end{array}$$

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Similarly, this is one of Keisler's Axioms for $Q^{>\omega}$:

$$\forall x[A(x) \to B(x)] \to (Q^{>\omega}x[A(x)] \to (Q^{>\omega}x[B(x)]))$$

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$$\forall x[A(x) \to B(x)] \to (Q^{>\omega}x[A(x)] \to (Q^{>\omega}x[B(x)]))$$

In other terms, if $\dfrac{A(x)}{B(x)}$ is a valid inference, then so are :

$$\frac{Most_x\,[C(x), A(x)]}{Most_x\,[C(x), B(x)]} \quad \text{and} \quad \frac{Q^{>\omega}x\,[A(x)]}{Q^{>\omega}x\,[B(x)]}$$

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Deep Inference for Generalized Quantifiers

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Alternatively, if you see logics as categories, they are the quantifiers that are *functors*.

Monotone quantifiers

In a model, logical consequence becomes subset inclusion: entailment-preserving quantifiers are exactly *monotone quantifiers*.

Definition

- $\mathcal{Q}$ is *monotone increasing* in an argument i if for all $(A_1, ..., A_n)$ and $B \supseteq A_i$, $(A_1, ..., A_i, ..., A_n) \in [\![\mathcal{Q}]\!]$ implies $(A_1, ..., B_i, ..., A_n) \in [\![\mathcal{Q}]\!]$.
- $\mathcal{Q}$ is *monotone decreasing* in an argument i if for all $(A_1, ..., A_n)$ and $B \subseteq A_i$, $(A_1, ..., A_i, ..., A_n) \in [\![\mathcal{Q}]\!]$ implies $(A_1, ..., B_i, ..., A_n) \in [\![\mathcal{Q}]\!]$

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Increasing quantifiers preserve derivations, and decreasing quantifiers "invert" derivations by switching premiss and conclusion.

Increasing quantifiers are functors, decreasing quantifiers are contravariant functors.

Quantifier Polarity

Definition

For a given argument i, we say that a quantifier $\mathcal{Q}$ is:

1. i-*positive* if for all φ_j, $\mathcal{Q}\overrightarrow{x}[\ldots, \varphi_{i-1}, \top, \varphi_{i+1}, \ldots] \equiv \top$.
2. i-*negative* if for all φ_j, $\mathcal{Q}\overrightarrow{x}[\ldots, \varphi_{i-1}, \bot, \varphi_{i+1}, \ldots] \equiv \top$.
3. i-*anti-positive* if for all φ_j, $\mathcal{Q}\overrightarrow{x}[\ldots, \varphi_{i-1}, \top, \varphi_{i+1}, \ldots] \equiv \bot$.
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Polarity essentially describes how quantifiers behave on $\top$ and $\bot$.

Useful to turn derivations into proofs, similar to the usual equalities:

$$\forall x.\top = \top = \exists x.\top$$

System QC

We give a formalization of the *Quantifier Calculus* QC, for monotone/polarized quantifiers:

- Built on the formalism of *Open Deduction*.
- Monotonicity allows inferences within quantifiers.
- Polarity gives us axioms to close proofs.

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For brevity, we present a version with unary (type $\langle k \rangle$) quantifiers, but the n-ary case is similar.

Formulae and Derivations

Usual first-order language, extended with monotone quantifiers $\mathcal{L}(\mathcal{Q}_1, \mathcal{Q}_2, ...)$. Formulae $\mathcal{F}$ and terms t are defined by:

$$t ::= x \mid c \mid f(t, ..., t)$$

$$\mathcal{F} = \top \mid \bot \mid p(t, ..., t) \mid \overline{p}(t, ..., t) \mid \mathcal{F} \vee \mathcal{F} \mid \mathcal{F} \wedge \mathcal{F} \mid \mathcal{Q}\overrightarrow{x}[\mathcal{F}]$$

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Negation as De Morgan dualities:

$$\overline{\top} = \bot \qquad \overline{\bot} = \top \qquad \overline{p(t_1, ..., t_n)} = \overline{p}(t_1, ..., t_n)$$

$$\overline{A \wedge B} = \overline{A} \vee \overline{B} \qquad \overline{A \vee B} = \overline{A} \wedge \overline{B} \qquad \overline{\mathcal{Q}\overrightarrow{x}[A]} = \mathcal{Q}^d\overrightarrow{x}[\overline{A}]$$

where the *dual* $\mathcal{Q}^d$ is defined as $\mathcal{Q}^d\overrightarrow{x}[A] = \overline{\mathcal{Q}\overrightarrow{x}[\overline{A}]}$.

Equational Theory of QC

$$A \vee \bot = A \qquad A \wedge \top = A \qquad \top \vee \top = \top \qquad \bot \wedge \bot = \bot$$

$$A \wedge B = B \wedge A \qquad A \vee B = B \vee A$$

$$A \wedge (B \wedge C) = (A \wedge B) \wedge C \qquad A \vee (B \vee C) = (A \vee B) \vee C$$

$\mathcal{Q}\overrightarrow{x}[\top] = \top$ if $\mathcal{Q}$ is positive $\qquad \mathcal{Q}\overrightarrow{x}[\bot] = \top$ if $\mathcal{Q}$ is negative

$\mathcal{Q}\overrightarrow{x}[\top] = \bot$ if $\mathcal{Q}$ is anti-positive $\qquad \mathcal{Q}\overrightarrow{x}[\bot] = \bot$ if $\mathcal{Q}$ is anti-negative

Inference Rules of QC

$$\text{i}\!\downarrow \frac{\top}{A \vee \overline{A}} \qquad \text{i}\!\uparrow \frac{A \wedge \overline{A}}{\bot}$$

$$\text{w}\!\downarrow \frac{\bot}{A} \qquad \text{w}\!\uparrow \frac{A}{\top}$$

$$\text{c}\!\downarrow \frac{A \vee A}{A} \qquad \text{c}\!\uparrow \frac{A}{A \wedge A}$$

$$\text{s}\,\frac{(A \vee B) \wedge C}{A \vee (B \wedge C)}$$

$$= \frac{A}{B}$$

if $A = B$

$$\text{qs}\,\frac{\mathcal{Q}\overrightarrow{x}[A(\overrightarrow{x}) \vee B] \wedge C}{\mathcal{Q}\overrightarrow{x}[A(\overrightarrow{x}) \wedge C] \vee B}$$

$$\text{qmix}\,\frac{\mathcal{Q}\overrightarrow{x}[A(\overrightarrow{x}) \wedge B] \wedge \overline{C}}{\mathcal{Q}\overrightarrow{x}[A(\overrightarrow{x}) \vee C] \vee \overline{B}}$$

Derivations

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- If $\begin{array}{c} A \\ {\scriptstyle\Pi}\,\| \\ B \end{array}$, $\begin{array}{c} C \\ {\scriptstyle\Delta}\,\| \\ D \end{array}$ are derivations and ${\scriptstyle\rho}\,\dfrac{B}{C}$ is a rule, then ${\scriptstyle\rho}\,\dfrac{\Pi}{\Delta}$ is a derivation $\begin{array}{c} A \\ \| \\ D \end{array}$.

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- If $\mathcal{Q}^{\uparrow}$ is an increasing quantifier and $\begin{array}{c} A \\ \| \\ B \end{array}$ is a derivation, then $\mathcal{Q}^{\uparrow}\overrightarrow{x}\begin{bmatrix} A \\ \| \\ B \end{bmatrix}$ is a derivation with premiss $\mathcal{Q}^{\uparrow}\overrightarrow{x}[A]$ and conclusion $\mathcal{Q}^{\uparrow}\overrightarrow{x}[B]$.

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- If $\mathcal{Q}^{\downarrow}$ is a decreasing quantifier and $\begin{array}{c} A \\ \| \\ B \end{array}$ is a derivation, then $\mathcal{Q}^{\downarrow}\overrightarrow{x}\begin{bmatrix} A \\ \| \\ B \end{bmatrix}$ is a derivation with premiss $\mathcal{Q}^{\downarrow}\overrightarrow{x}[B]$ and conclusion $\mathcal{Q}^{\downarrow}\overrightarrow{x}[A]$.

Soundness

Theorem

If there is a derivation $\begin{matrix} A \\ \| \\ B \end{matrix}$ *, then* $A \models B$*. In particular, if a formula is provable then it is valid.*

Proof.

By induction on derivations. The soundness of the composition of derivations by a quantifier is a consequence of monotonicity. □

Examples

For any *increasing* quantifier $\mathcal{Q}^{\uparrow}$, the rule $\text{qc}\uparrow \dfrac{\mathcal{Q}^{\uparrow}\overrightarrow{x}[A(\overrightarrow{x}) \wedge B(\overrightarrow{x})]}{\mathcal{Q}^{\uparrow}\overrightarrow{x}[A(\overrightarrow{x})] \wedge \mathcal{Q}^{\uparrow}\overrightarrow{x}[B(\overrightarrow{x})]}$ is admissible:

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$$\mathsf{c}{\uparrow}\ \dfrac{\mathcal{Q}\overrightarrow{x}[A(\overrightarrow{x}) \wedge B(\overrightarrow{x})]}{\mathcal{Q}^{\uparrow}\overrightarrow{x}\left[{=}\,\dfrac{A(\overrightarrow{x}) \wedge \boxed{\mathsf{w}{\uparrow}\ \dfrac{B(\overrightarrow{x})}{\top}}}{A(\overrightarrow{x})}\right] \wedge \mathcal{Q}^{\uparrow}\overrightarrow{x}\left[{=}\,\dfrac{\boxed{\mathsf{w}{\uparrow}\ \dfrac{A(\overrightarrow{x})}{\top}} \wedge B(\overrightarrow{x})}{B(\overrightarrow{x})}\right]}$$

Examples

For any *decreasing* quantifier $\mathcal{Q}^{\downarrow}$, the rule $\mathsf{dqc}{\downarrow}\, \dfrac{\mathcal{Q}^{\downarrow}\overrightarrow{x}[A(\overrightarrow{x}) \vee B(\overrightarrow{x})]}{\mathcal{Q}^{\downarrow}\overrightarrow{x}[A(\overrightarrow{x})] \wedge \mathcal{Q}^{\downarrow}\overrightarrow{x}[B(\overrightarrow{x})]}$ is admissible:

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\mathsf{c}{\uparrow}\ \dfrac{\mathcal{Q}^{\downarrow}\overrightarrow{x}[A(\overrightarrow{x}) \vee B(\overrightarrow{x})]}{\mathcal{Q}^{\downarrow}\overrightarrow{x}\left[= \dfrac{A(\overrightarrow{x})}{A(\overrightarrow{x}) \vee \boxed{\mathsf{w}{\downarrow}\ \dfrac{\perp}{B(\overrightarrow{x})}}}\right] \wedge \mathcal{Q}^{\downarrow}\overrightarrow{x}\left[= \dfrac{B(\overrightarrow{x})}{\boxed{\mathsf{w}{\downarrow}\ \dfrac{\perp}{A(\overrightarrow{x})}} \vee B(\overrightarrow{x})}\right]}
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Remark that since $\mathcal{Q}^{\downarrow}$ is decreasing, the inner derivations are "upside-down".

Adding specific rules

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If there is the need, we can add to QC any rule $\dfrac{F}{G}$, under the condition that it is *locally sound*, i.e. for all $\mathcal{M}$, $\mathcal{M} \models F$ implies $\mathcal{M} \models G$.

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If there is the need, we can add to QC any rule $\dfrac{F}{G}$, under the condition that it is *locally sound*, i.e. for all $\mathcal{M}$, $\mathcal{M} \models F$ implies $\mathcal{M} \models G$.

Examples of rules that are globally sound, but not locally sound are typically rules with side conditions:

$$\forall\,\frac{A(x)}{\forall x A(x)}\text{, with } x \text{ eigenvariable}$$

Example

For example, we can reintroduce the instanciation rule for $\exists$: $\mathsf{n}{\downarrow}\,\dfrac{A(t)}{\exists x A(x)}$ and use it within other quantifiers:

$$
\mathsf{qs}\,\dfrac{=\,\dfrac{\top}{\mathcal{Q}^{\uparrow}y\left[\mathsf{i}{\downarrow}\,\dfrac{\top}{A(y) \vee \boxed{\mathsf{n}{\downarrow}\,\dfrac{\overline{A}(y)}{\exists x \overline{A}(x)}}}\right]}}{\exists x \overline{A}(x) \vee \mathcal{Q}^{\uparrow}y[A(y)]}
\qquad
\mathsf{qmix}\,\dfrac{=\,\dfrac{\top}{\mathcal{Q}^{\downarrow}y\left[\mathsf{i}{\uparrow}\,\dfrac{\overline{\exists x A(x)} \wedge \boxed{\mathsf{n}{\downarrow}\,\dfrac{A(y)}{\exists x A(x)}}}{\bot}\right]}}{\exists x A(x) \vee \mathcal{Q}^{\downarrow}y[A(y)]}
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n-ary Quantifiers

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The rules qs and qmix are also sound in their n-ary version:

$$\text{qs}\ \frac{\mathcal{Q}\overrightarrow{x}[\ldots, A(\overrightarrow{x}) \vee B, \ldots] \wedge C}{\mathcal{Q}\overrightarrow{x}[\ldots, A(\overrightarrow{x}) \wedge C, \ldots] \vee B} \qquad \text{qmix}\ \frac{\mathcal{Q}\overrightarrow{x}[\ldots, A(\overrightarrow{x}) \wedge B, \ldots] \wedge \overline{C}}{\mathcal{Q}\overrightarrow{x}[\ldots, A(\overrightarrow{x}) \vee C, \ldots] \vee \overline{B}}$$

n-ary Quantifiers

Example: $More_x[A, B]$, with $[\![More]\!] = \{(A, B) \mid |A| \geq |B|\}$
$More$ is 1-increasing, 2-decreasing, and for any A, $More_x[A(x), A(x)] = \top$.

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\mathsf{qmix}\,\dfrac{\mathsf{qs}\,\dfrac{=\,\dfrac{\top}{More_x\left[\mathsf{s}\,\dfrac{=\,\dfrac{A(x)}{\boxed{\mathsf{i}{\downarrow}\,\dfrac{\top}{B(x) \vee \overline{B}(x)}} \wedge A(x)}}{B(x) \vee \boxed{\mathsf{n}{\downarrow}\,\dfrac{A(x) \wedge \overline{B}(x)}{\exists y[A(y) \wedge \overline{B}(y)]}}}\;,\;\mathsf{s}\,\dfrac{C(x) \wedge \boxed{\mathsf{n}{\uparrow}\,\dfrac{\forall y[\overline{C}(y) \vee A(y)]}{\overline{C}(x) \vee A(x)}}}{=\,\dfrac{\boxed{\mathsf{i}{\uparrow}\,\dfrac{C(x) \wedge \overline{C}(x)}{\bot}} \vee A(x)}{A(x)}}\right]}}{\exists y[A(y) \wedge \overline{B}(y)] \vee More_x\left[B(x), C(x) \wedge \forall y[\overline{C}(y) \vee A(y)]\right]}}{\exists y[C(y) \wedge \overline{A}(y)] \vee \exists y[A(y) \wedge \overline{B}(y)] \vee More_x\left[B(x), C(x)\right]}
$$

is a proof of:

$$\forall y[C(y) \to A(y)] \wedge \forall y[A(y) \to B(y)] \to More_x\left[B(x), C(x)\right]$$

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- Quantifiers "more than κ", "less than κ" for any cardinal κ, strict or not.
- The majority of natural language quantifiers are monotone in at least one argument, e.g. "most", "more ... than ...", "almost all", "many", "few", etc.
- Conjunctions and disjunctions of increasing (resp. decreasing) quantifiers are also increasing (resp. decreasing). Negation reverses monotonicity.

Perspectives

- QC is a baseline sound proof system for the broadest class of quantifiers suitable for deep inference: monotone quantifiers.

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- It can be seamlessly adapted to more specific cases by adding ad-hoc rules expressing particular properties.

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- Do usual structural properties hold ?
  - Locality
  - Cut-elimination/normalization
  - Decomposition theorems

Thank you for your attention !