First-order Intuitionistic Multiplicative Exponential Linear Logic with Hilbert's Epsilon

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Introduction

- The Epsilon Calculus: introduced by Hilbert to study proofs and consistency of logical theories.

- More recently, a renewed interest for its interesting properties in classical first-order proof theory, such as compression of cut-free proofs.

- This (ongoing) work aims to study in what capacity the ε-calculus can be integrated into Linear Logic, as a novel approach to LL quantification.

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or in sequent-style:

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  $\rightsquigarrow$ (barely) disguised injection of the Axiom of Choice in the syntax.

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- And similarly we get $\forall x A(x) \equiv A(\varepsilon_x \neg A)$.

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- ε-translation of quantified formulae:

$$\forall x \exists y P(x, y)$$

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The ε-theorems

First ε-theorem

If E is a formula without ε-terms, and if there is an ε-proof of E, then there is a proof of E containing no critical ε-rule.

$\leadsto$ Conservativity over the elementary calculus.

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Extended First ε-theorem

If $E(e_1, \dots, e_m)$ is a (quantifier-free) formula containing only the ε-terms $e_1, \dots, e_m$, and if there is a proof of $E(e_1, \dots, e_m)$, then there are ε-free terms t^i_j, $1 \leq i \leq n, 1 \leq j \leq m$ and a proof of $\bigvee_{i=1}^{n} E(t^i_1, \dots, t^i_m)$ containing no critical ε-rule.

$\leadsto$ Analogue to Herbrand's Theorem for ε-calculi.

Non-elementary proof compression

- The ε-calculus admits shorter cut-free proofs than traditional sequent calculus.

Baaz, Lolić, 2024

There are sequents for which the size of their smallest cut-free LK proof is not bounded by any elementary function of the size of ther smallest cut-free ε-proof.

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- Example: shortest cut-free proof of the Drinkers' Formula

$$\mathit{cont}\,\dfrac{\exists\,\dfrac{\vee\,\dfrac{\forall\,\dfrac{\exists\,\dfrac{\vee\,\dfrac{\mathit{ax}\,\dfrac{}{\vdash \neg P(t), P(z), \neg P(z), \forall y P(y)}}{\vdash \neg P(t), P(z), \neg P(z) \vee \forall y P(y)}}{\vdash \neg P(t), P(z), \exists x(\neg P(x) \vee \forall y P(y))}}{\vdash \neg P(t), \forall y P(y), \exists x(\neg P(x) \vee \forall y P(y))}}{\vdash \neg P(t) \vee \forall y P(y), \exists x(\neg P(x) \vee \forall y P(y))}}{\vdash \exists x(\neg P(x) \vee \forall y P(y)), \exists x(\neg P(x) \vee \forall y P(y))}}{\vdash \exists x(\neg P(x) \vee \forall y P(y))}$$

$$\varepsilon_R\,\dfrac{\mathit{ax}\,\dfrac{}{\vdash \neg P(\varepsilon_y \neg P) \vee P(\varepsilon_y \neg P)}}{\vdash \neg P(\varepsilon_x(\neg P(x) \vee P(\varepsilon_y \neg P)) \vee P(\varepsilon_y \neg P))}$$

Linear Logic with ε

- Several reasons to consider adding ε to Linear Logic:
    - Leverage the shorter proofs provided by the ε-calculus.
    - Treat quantification in multiplicative LL without the inherently additive $\forall, \exists$.
    - Also a way to get rid of quantifier boxes in proof nets (more on that later).

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- For now, we restrict ourselves to Intuitionistic Multiplicative (Exponential) Linear Logic (IM(E)LL).

IMELL + ε

- IMELLε terms t and formulae $\mathcal{F}$ are defined conjointly:

$$t := x \mid c \mid f(t, ..., t) \mid \varepsilon_x \mathcal{F}$$

$$\mathcal{F} := \mathbf{1} \mid \bot \mid P(t, ..., t) \mid \mathcal{F} \otimes \mathcal{F} \mid \mathcal{F} \multimap \mathcal{F} \mid !\mathcal{F}$$

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- Sequent calculus: same rules as traditional IMELL, but quantifier rules are replaced by ε-rules.

$$\varepsilon_L \frac{\Gamma, A(x) \vdash C}{\Gamma, A(\varepsilon_x A) \vdash C} \; x \notin \mathit{free}(\Gamma) \qquad \varepsilon_R \frac{\Gamma \vdash A(t)}{\Gamma \vdash A(\varepsilon_x A)}$$

Hyperdoctrines

- Usual semantics of the classical ε-calculus: classical models with choice functions.

$$\mathcal{M}, \varphi \models A(\varepsilon_x B) \quad \text{iff} \quad \varphi([\![B]\!]^{\mathcal{M}}) \in [\![A]\!]^{\mathcal{M}}$$

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First-order Hyperdoctrine

A **first-order hyperdoctrine** *is a couple* (C, P) *where:*

- C *is a category with finite products.*
- $P : C^{op} \to \textbf{Pos}$ *is a contravariant functor to the category of posets.*

Hyperdoctrines

Given a first-order language $\mathcal{L}$, we can define the category $Ctx_{\mathcal{L}}$ of $\mathcal{L}$-contexts, in which:

- Objects are finite sequences of distincts variables $(x_1, ..., x_n)$

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$$[t_1(\overrightarrow{x})/y_1, ..., t_n(\overrightarrow{x})/y_n]$$
- Composition is the composition of term substitutions.
- The identity of $\overrightarrow{x}$ is the identity substitution $[x_1/x_1, ..., x_n/x_n]$.

Hyperdoctrines

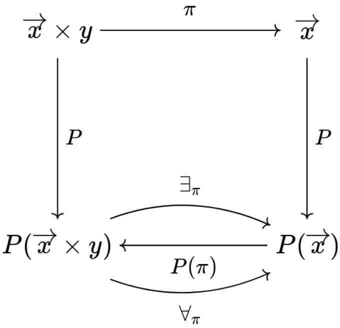
If $P : Ctx_{\mathcal{L}}^{op} \to \mathbf{Pos}$ is a contravariant functor, then $(Ctx_{\mathcal{L}}, P)$ is a hyperdoctrine:

- Products and projections in $Ctx_{\mathcal{L}}$ are context products/projections.
- Each $P(\overrightarrow{x})$ can be seen as the collection of $\mathcal{L}$-formulae over the context $\overrightarrow{x}$, ordered by entailment: $\phi(\overrightarrow{x}) \leq \psi(\overrightarrow{x})$ iff $\phi(\overrightarrow{x}) \vdash \psi(\overrightarrow{x})$ is provable.

$$\begin{array}{ccc}
\overrightarrow{x} \times y & \xrightarrow{\quad\pi\quad} & \overrightarrow{x} \\
\Big\downarrow P & & \Big\downarrow P \\
P(\overrightarrow{x} \times y) & \xleftarrow[\quad P(\pi)\quad]{} & P(\overrightarrow{x})
\end{array}$$

Hyperdoctrines

$(Ctx_{\mathcal{L}}, P)$ is said to be ***existential*** (resp. ***universal***) if for every projection π, $P(\pi)$ has a left adjoint $\exists\pi$ (resp. right adjoint $\forall\pi$):



such that:

- Quantification commutes with substitution (***Beck-Chevalley condition***)
- For every $\phi \in P(\overrightarrow{x} \times y)$ and $\psi \in P(\overrightarrow{x})$, we have: $\exists_\pi(P(\pi)(\psi) \wedge \phi) = \exists_\pi(\phi) \wedge \psi$ (***Frobenius reciprocity***).

Hyperdoctrines

Additional structure of the posets provide categorical semantics for various first-order logics, depending on the internal logic of each $P(\overrightarrow{x})$ (when seen as categories):

- Heyting algebras: Intuitionistic first-order logic.
- Boolean algebras: Classical first-order logic.
- Star-autonomous categories: First-order Multiplicative Linear Logic.
- etc.

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What structure for ε-calculi ?

ε-hyperdoctrines

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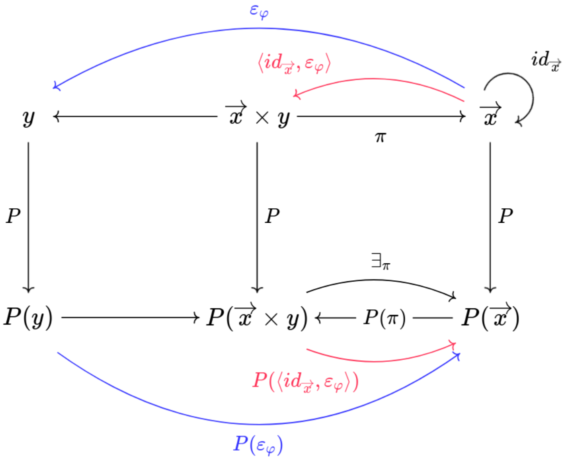
ε-hyperdoctrine

$(Ctx_{\mathcal{L}}, P)$ is said to have the ε-operator if for every $\overrightarrow{x}, y$ and every $\varphi \in P(\overrightarrow{x} \times y)$, there is a morphism $\varepsilon_\varphi : \overrightarrow{x} \to y$ such that:

$$\exists_\pi(\varphi) = P(\langle id_{\overrightarrow{x}}, \varepsilon_\varphi \rangle)(\varphi)$$

ε-hyperdoctrines



$$\exists_\pi(\varphi) = P(\langle id_{\vec{x}}, \varepsilon_\varphi \rangle)(\varphi) \quad \text{means} \quad \exists x \varphi(x) \equiv \varphi(\varepsilon_x \varphi(x))$$

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- The internal language of the posets $P(\overrightarrow{x})$ must be IMLL, i.e. they must be symmetric monoidal closed categories.
- An interpretation of the exponential "!", i.e. a comonad on $P(\overrightarrow{x})$.
- And an interpretation of ε-terms.

So we just have to find the right structure for the truth values/interpretations of formulae in $P(\overrightarrow{x})$.

Truth value ordered monoid

- Define $M = (\{0\} \cup \{\frac{1}{2^n} \mid n \in \mathbb{N}\}, \cdot, \leq)$, with $\cdot, \leq$ the standard multiplication and order on $\mathbb{Q}$.

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  - $[\![A \multimap B]\!] = \begin{cases} 1 & \text{if } [\![A]\!] = 0 \\ min(1, [\![B]\!]/[\![A]\!]) & \text{otherwise} \end{cases}$

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- But then the interpretation of "!" is too trivial.

- We simply take M^ω, ω copies of M for truth values, with $\cdot, \leq, !$ defined pointwise on M^ω.

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- Every non-empty subset of M^ω has a maximal element, so we simply take:

$$[\![P(\langle id_{\vec{x}}, \varepsilon_A\rangle)(A)]\!] = \max_\pi([\![\exists_\pi(A)]\!])$$

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In summary:

- Every ε-hyperdoctrine in which for each poset $P(\overrightarrow{x})$, the formulae over the context $\overrightarrow{x}$ are assigned truth values in the ordered monoid M^{ω}, provides an interpretation for IMELL + ε.

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- The semantics proposed are semantics of ***formulae***, not semantics of ***proofs***.
- We work with intuitionistic linear logic to avoid issues related to negation, double-negation closure, etc.

Classical MELL + ε

- As defined here, IMELL ε-hyperdoctrines do not accomodate for linear negation $(.)^{\perp}$, and thus for universal quantification in the form $\forall x A(x) \equiv A(\varepsilon_x A^{\perp})$.

Classical MELL + ε

- As defined here, IMELL ε-hyperdoctrines do not accomodate for linear negation $(.)^\perp$, and thus for universal quantification in the form $\forall x A(x) \equiv A(\varepsilon_x A^\perp)$.

- One could define $A^\perp$ as $A \multimap \perp$, but in this case this is not fine-grained enough, since $A^\perp$ always collapses to $\perp$ unless $[\![A]\!] = 0$.

- Future investigations: provide similar categorical semantics for classical MELL + ε.

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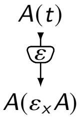

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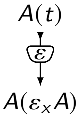

- No eigenvariable conditions $\rightsquigarrow$ no quantifier boxes.

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Thank you for your attention !