

Generalized Quantifiers and Prenex Normal Forms

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Abstract

Generalized quantifiers, introduced by Mostowski and Lindström, are generalizations of the standard quantifiers of modern logic, \forall and \exists , and have been extensively studied for their applications to model theory and computational linguistics, but not so much for their purely syntactical and proof-theoretic properties. In this paper, we study under which conditions first-order logics extended with generalized quantifiers have prenex normal form theorems. We introduce the notion of polarity of a quantifier, and show that it provides a sufficient condition for the existence of syntactic transformation rules. We establish a prenex normal form theorem for first-order languages whose quantifiers have the adequate polarity properties, and show that some well-known classes of quantifiers, namely monotone quantifiers and Henkin quantifiers, fall under this category.

Keywords: Logic, Proof Theory, Generalized Quantifiers, Prenex Normal Forms

1 Introduction

Generalized quantifiers, initially introduced by Mostowski (1957) and further developed by Lindström (1966) are a generalization of standard first-order quantification that treats quantifiers as generic relations over relations (or tuples of relations) on the domain (a.k.a universe, model) of an interpretation: thus, quantifiers are viewed as higher-order concepts.

The concept of generalized quantifiers is not new, and has been extensively studied from a model-theoretic and linguistic viewpoint. This is not surprising, as it originates in the generalization of the usual quantifiers' interpretation in a given model, to generic relations on this model. The expressive power of this formalism has also found a significant use in linguistics, for its ability to account for the great diversity of quantification in natural language. However, very little work has been done on

the more syntactic and proof-theoretic aspects of generalized quantifiers, which can be explained by their expressive power: finding sensible syntactic properties and reasonable deduction systems for non-trivial quantifier classes is *in general* considerably more challenging. Among the few works on proof theory for some very limited subsets of quantifiers, we can cite [Keisler \(1970\)](#) on quantifiers such as "uncountably many", and more recently the works of [Baaz and Lolić \(2021\)](#) on Henkin quantifiers.

In a very broad sense, the general aim of our work is to continue these proof-theoretical approaches of quantifiers, and if it is most likely impossible for the full theory of generalized quantifiers, to find classes of quantifiers for which we can have reasonable properties and deduction systems. In this article specifically, we study the possible existence of prenex normal forms for languages with general quantifiers. The concept of prenex normal forms (i.e. putting quantifiers at the start of the formula), is an important syntactic property for logical languages and is widely used in automated theorem proving, which makes it an interesting question if we want to develop such systems for generalized quantifiers.

2 A Reminder on Generalized Quantifiers

We will start by giving here a very succinct presentation of generalized quantifiers for the needs of this paper, mainly inspired by [Peters and Westerståhl \(2006\)](#) and Chapter 10 of [Väänänen \(2011\)](#). [Szymanik \(2016\)](#) also gives another very complete overview of the topic. We refer the reader to these works for a more thorough description of the wide subject of generalized quantifiers.

2.1 Basic Definitions

Definition 2.1 (Weak Generalized Quantifier) *A weak (generalized) quantifier of type $\langle k_1, \dots, k_n \rangle$ is a mapping $Q(M)$ from each non-empty set M to a subset of $\mathcal{P}(M)^{k_1} \times \dots \times \mathcal{P}(M)^{k_n}$.*

In other words, a quantifier on a set M is simply a n -ary relation over k_i -ary relations on M .

Examples:

1. The type $\langle 1 \rangle$ existential quantifier \exists is the mapping:

$$\exists(M) = \{A \subseteq M \mid A \neq \emptyset\}$$

2. The type $\langle 1 \rangle$ quantifier "infinitely many" is:

$$Q^{\geq \omega}(M) = \{A \subseteq M \mid A \text{ is infinite}\}$$

3. The type $\langle 1, 1 \rangle$ quantifier "more ... than ..." is the binary relation:

$$More(M) = \{(A, B) \in \mathcal{P}(M) \times \mathcal{P}(M) \mid |A| > |B|\}$$

Definition 2.2 (Negations, Duals) *For any quantifier Q , we define:*

- The outer negation of Q as the quantifier:

$$(\neg Q)(M) = \{(A_1, \dots, A_n) \in \mathcal{P}(M)^{k_1} \times \dots \times \mathcal{P}(M)^{k_n} \mid (A_1, \dots, A_n) \notin Q(M)\}$$

- The inner negation of Q on the i -th argument as:

$$(Q\neg_i)(M) = \{(A_1, \dots, A_i, \dots, A_n) \in \mathcal{P}(M)^{k_1} \times \dots \times \mathcal{P}(M)^{k_n} \mid (A_1, \dots, \overline{A_i}, \dots, A_n) \in Q(M)\}$$

where $\overline{A_i}$ is the complement of A_i in $\mathcal{P}(M)^{k_i}$.

- The dual of Q on the i -th argument as the quantifier $Q^{d_i} = (\neg Q)\neg_i = \neg(Q\neg_i)$.

We can remark here that a n -ary quantifier has therefore n possibly different duals, and in case $n = 1$ we recover the usual notion of quantifier negation and duality.

2.2 First-order Logic with Generalized Quantifiers

The notion of generalized quantifier as defined above is a purely model-theoretic one. However, we will show that for any generalized quantifier, we can define a new corresponding logical operator extending a chosen logical language.

In the remainder of this work, we will place ourselves within the framework of the usual language of first-order logic, which we denote \mathcal{L} . There is of course also a sensible notion of generalized quantifiers extending other logics, for example second-order logic, but this is much beyond the point of this paper.

We consider the language of first-order logic, with terms t and formulas \mathcal{F} defined by the grammars:

$$\begin{aligned} t &:= c \mid x \mid f(t, \dots, t) \\ \mathcal{F} &:= \top \mid \perp \mid P(t, \dots, t) \mid \neg \mathcal{F} \mid \mathcal{F} \vee \mathcal{F} \mid \mathcal{F} \wedge \mathcal{F} \mid \forall x \mathcal{F} \mid \exists x \mathcal{F} \end{aligned}$$

with symbols c for constants, x for variables, f for functions and P for predicates.

We write implication $\mathcal{F} \rightarrow \mathcal{G}$ as a shorthand for $\neg \mathcal{F} \vee \mathcal{G}$. We include \forall, \exists , as it is the usual definition of the language, but it is also possible to consider them just as simply specific examples of generalized quantifiers. In fact, in the rest of this paper, we do not distinguish and speak of "quantifiers" to mean generalized quantifiers as well as \forall, \exists .

A model $\mathcal{M} = (M, I, s)$ is an usual \mathcal{L} -structure, with domain M , an interpretation I mapping constants to elements of M , n -ary function symbols to functions $M^n \rightarrow M$ and n -ary predicates to n -ary relations on M , and with a free variable assignment s , mapping variables to elements of M . For a formula φ , we denote by $\varphi[\bar{a}/\bar{x}]$ the formula obtained by substituting in φ every free occurrence of the variables \bar{x} by the terms \bar{a} .

To each generalized quantifier in the set-theoretic sense, we can associate quite naturally a corresponding logical quantifier symbol extending \mathcal{L} .

Definition 2.3 *For a type $\langle k_1, \dots, k_n \rangle$ quantifier Q , we add to \mathcal{L} a new quantifier symbol \mathcal{Q} , and the following formula formation rule:*

- If $\bar{x}_1, \dots, \bar{x}_n$ are respectively k_1, \dots, k_n -tuples of variables and $\varphi_1, \dots, \varphi_n$ are formulas, then:

$$Q\bar{x}_1, \dots, \bar{x}_n[\varphi_1, \dots, \varphi_n]$$

is a formula, in which for every i the variables \bar{x}_i are bound in φ_i by Q .

The semantics of this new logical symbol is then given by the corresponding quantifier Q , that is, in a model \mathcal{M} with domain M we simply interpret Q by $Q(M)$:

$$\mathcal{M} \models Q\bar{x}_1, \dots, \bar{x}_n[\varphi_1, \dots, \varphi_n]$$

iff

$$(\{(\bar{a}_1) \in M^{k_1} \mid \mathcal{M} \models \varphi_1[\bar{a}_1/\bar{x}_1]\}, \dots, \{(\bar{a}_n) \in M^{k_n} \mid \mathcal{M} \models \varphi_n[\bar{a}_n/\bar{x}_n]\}) \in Q(M)$$

We note $\mathcal{L}(Q_1, Q_2, \dots)$ the language of first-order logic augmented with the corresponding quantifier symbols. Q is sometimes called a *global* quantifier, and $Q(M)$ a *local* quantifier. For a given, fixed Q , and for the sake of readability, we will often allow ourselves to abuse notation and write $Q\bar{x}[\dots]$ for $Q\bar{x}[\dots]$.

By this definition, it is immediate to see that the definitions and properties of negation, inner negation and duals for local quantifiers naturally lift to the corresponding global quantifiers.

A naturally occurring question is whether we should automatically include negations and dual quantifiers in the extension of the language. In general, there is no obligation that the language $\mathcal{L}(Q)$ contains the dual quantifiers of Q , but more often than not it is practical to include them as well. In particular, this would allow one to recover the familiar duality equivalence: $\neg Q\bar{x}[\dots, \varphi_i, \dots] \equiv Q^{d_i}\bar{x}[\dots, \neg\varphi_i, \dots]$. If all duals are part of the language, i.e. if for all $Q \in \mathcal{L}$, $Q^{d_i} \in \mathcal{L}$ for all i , we say that \mathcal{L} is *closed by quantifier duality*.

3 Quantifier i -polarities

In order to study prenex normal forms for generalized quantifiers, we need to introduce the notion of *polarity* of a quantifier in a given argument i .

Definition 3.1 Let Q be a quantifier of type $\langle k_1, \dots, k_n \rangle$. With respect to an argument i and a model \mathcal{M} , we say that Q is:

- i -positive in \mathcal{M} if for all $(A_1, \dots, A_i, \dots, A_n) \in \mathcal{P}(M)^{k_1} \times \dots \times \mathcal{P}(M)^{k_n}$:

$$(A_1, \dots, M^{k_i}, \dots, A_n) \in Q(M)$$

- i -anti-positive in \mathcal{M} if for all $(A_1, \dots, A_i, \dots, A_n) \in \mathcal{P}(M)^{k_1} \times \dots \times \mathcal{P}(M)^{k_n}$:

$$(A_1, \dots, M^{k_i}, \dots, A_n) \notin Q(M)$$

- *i*-negative in \mathcal{M} if for all $(A_1, \dots, A_i, \dots, A_n) \in \mathcal{P}(M)^{k_1} \times \dots \times \mathcal{P}(M)^{k_n}$:

$$(A_1, \dots, \emptyset, \dots, A_n) \in Q(M)$$

- *i*-anti-negative in \mathcal{M} if for all $(A_1, \dots, A_i, \dots, A_n) \in \mathcal{P}(M)^{k_1} \times \dots \times \mathcal{P}(M)^{k_n}$:

$$(A_1, \dots, \emptyset, \dots, A_n) \notin Q(M)$$

If we transpose this definition to the syntactic side, i.e. on the global quantifier associated with Q , we can see that polarity essentially describes how Q behaves with respect to \top and \perp , as stated by the following result.

Proposition 3.2 *Let Q be a quantifier of type $\langle k_1, \dots, k_n \rangle$. We have that:*

- If Q is *i*-positive in \mathcal{M} , then for all formulas $(\varphi_1, \dots, \varphi_n)$:

$$\mathcal{M} \models Q\bar{x}[\varphi_1, \dots, \top, \dots, \varphi_n]$$

- If Q is *i*-anti-positive in \mathcal{M} , then for all formulas $(\varphi_1, \dots, \varphi_n)$:

$$\mathcal{M} \not\models Q\bar{x}[\varphi_1, \dots, \top, \dots, \varphi_n]$$

- If Q is *i*-negative in \mathcal{M} , then for all formulas $(\varphi_1, \dots, \varphi_n)$:

$$\mathcal{M} \models Q\bar{x}[\varphi_1, \dots, \perp, \dots, \varphi_n]$$

- If Q is *i*-anti-negative in \mathcal{M} , then for all formulas $(\varphi_1, \dots, \varphi_n)$:

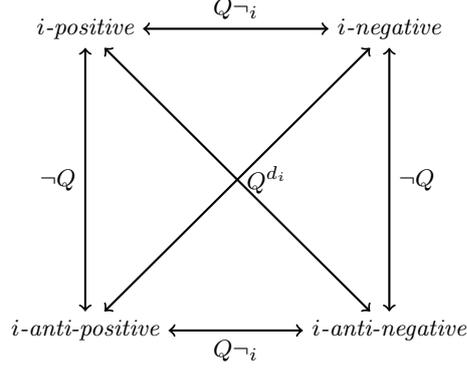
$$\mathcal{M} \not\models Q\bar{x}[\varphi_1, \dots, \perp, \dots, \varphi_n]$$

The converse, however, is only true with respect to *definable subsets* of \mathcal{M} . For example, if for all formulas $(\varphi_1, \dots, \varphi_n)$, $\mathcal{M} \models Q\bar{x}[\varphi_1, \dots, \top, \dots, \varphi_n]$, then for all definable A_1, \dots, A_n , we do have $(A_1, \dots, M^{k_i}, \dots, A_n) \in Q(M)$, but this is not necessarily true for *all* subsets A_1, \dots, A_n . For example, we could consider a type $\langle 1, 1 \rangle$ quantifier Q^\emptyset and a model $\mathcal{M} = (M, I)$ such that $Q^\emptyset(M) = \{(S, \emptyset)\}$ for some \mathcal{L} -undefinable subset S . Then for all \mathcal{L} -formulas φ , $\mathcal{M} \not\models Qx^\emptyset[\varphi, \perp]$, but Q^\emptyset is not 1-anti-negative in \mathcal{M} .

The usual quantifiers \forall, \exists are both positive ($\exists x \top \equiv \forall x \top \equiv \top$) and anti-negative ($\exists x \perp \equiv \forall x \perp \equiv \perp$).

A first immediate useful property is that inner and outer negation (and thus duality) invert the polarization of a quantifier, in the way given by the next lemma.

Lemma 3.3 *For any quantifier Q , we have the following square of oppositions:*



Proof The proof is immediate by the definitions of polarities and negations. Essentially, outer negation takes the complement of the quantifier, so it transforms i -polarities into i -anti-polarities and conversely, and inner negation on i takes the complement of the i -th argument subset, so it transforms i -*-positive polarities into i -*-negative ones and conversely. Duality is the composition of both. \square

It is also worth noting that while a quantifier cannot be positive *and* anti-positive (nor negative *and* anti-negative) in the same argument, it can be both positive and negative (or anti-positive and anti-negative) in the same argument, and in general, i -polarity is independent of j -polarity for $i \neq j$.

If a quantifier Q is i -positive in \mathcal{M} for *all* models \mathcal{M} , we will simply say that Q is i -positive, and similarly for the other polarities. In the same spirit, we extend the notion to the global quantifiers, and say that \mathcal{Q} is i -positive if Q is i -positive, and so on.

For quantifiers that have polarities, we can establish quantifier shift rules, as follows.

Lemma 3.4 *Let Q be a quantifier, \bar{x} variables, φ_i a formula with free variables \bar{x} , and ψ a formula which does not contain any of the variables \bar{x} . We have the equivalences:*

1. *If Q is positive in the i -th argument, then*

$$\mathcal{Q}\bar{x}[\dots, \varphi_i(\bar{x}) \vee \psi, \dots] \equiv \mathcal{Q}\bar{x}[\dots, \varphi_i(\bar{x}), \dots] \vee \psi$$

2. *If Q is anti-positive in the i -th argument, then*

$$\mathcal{Q}\bar{x}[\dots, \varphi_i(\bar{x}) \vee \psi, \dots] \equiv \mathcal{Q}\bar{x}[\dots, \varphi_i(\bar{x}), \dots] \wedge \neg\psi$$

3. *If Q is negative in the i -th argument, then*

$$\mathcal{Q}\bar{x}[\dots, \varphi_i(\bar{x}) \wedge \psi, \dots] \equiv \mathcal{Q}\bar{x}[\dots, \varphi_i(\bar{x}), \dots] \vee \neg\psi$$

4. If Q is anti-negative in the i -th argument, then

$$\mathcal{Q}\bar{x}[\dots, \varphi_i(\bar{x}) \wedge \psi, \dots] \equiv \mathcal{Q}\bar{x}[\dots, \varphi_i(\bar{x}), \dots] \wedge \psi$$

Proof We will only present here the proof for the first case (Q positive). The proof for the other cases is unremarkably similar, or can be obtained from this one by simply using the properties on the polarity of $\neg Q$ and $Q\neg_i$.

Let Q be a quantifier, positive in the i -th argument.

1. Let $\mathcal{M} = (M, I)$ be a model, and suppose that $\mathcal{M} \models \mathcal{Q}\bar{x}[\dots, \varphi_i(\bar{x}) \vee \psi, \dots]$.
 If $\mathcal{M} \models \psi$, then trivially $\mathcal{M} \models \mathcal{Q}\bar{x}[\dots, \varphi_i(\bar{x}), \dots] \vee \psi$.
 If $\mathcal{M} \not\models \psi$, then $\{\bar{x} \in M^{k_i} \mid \mathcal{M} \models \varphi_i(\bar{x}) \vee \psi\} = \{\bar{x} \in M^{k_i} \mid \mathcal{M} \models \varphi_i(\bar{x})\}$. Thus, from the hypothesis $\mathcal{M} \models \mathcal{Q}\bar{x}[\dots, \varphi_i(\bar{x}) \vee \psi, \dots]$, we have also $\mathcal{M} \models \mathcal{Q}\bar{x}[\dots, \varphi_i(\bar{x}), \dots]$, which means $\mathcal{M} \models \mathcal{Q}\bar{x}[\dots, \varphi_i(\bar{x}), \dots] \vee \psi$.
2. Let $\mathcal{M} = (M, I)$ be a model, and suppose that $\mathcal{M} \models \mathcal{Q}\bar{x}[\dots, \varphi_i(\bar{x}), \dots] \vee \psi$.
 If $\mathcal{M} \models \psi$, then $\{\bar{x} \in M^{k_i} \mid \mathcal{M} \models \varphi_i(\bar{x}) \vee \psi\} = \{\bar{x} \in M^{k_i} \mid \mathcal{M} \models \top\}$. Since Q is taken to be positive, we obtain that $\mathcal{M} \models \mathcal{Q}\bar{x}[\dots, \varphi_i(\bar{x}) \vee \psi, \dots]$.
 If $\mathcal{M} \not\models \psi$, then by hypothesis, necessarily $\mathcal{M} \models \mathcal{Q}\bar{x}[\dots, \varphi_i(\bar{x}), \dots]$. Since $\mathcal{M} \not\models \psi$, we also have $\{\bar{x} \in M^{k_i} \mid \mathcal{M} \models \varphi_i(\bar{x}) \vee \psi\} = \{\bar{x} \in M^{k_i} \mid \mathcal{M} \models \varphi_i(\bar{x})\}$. Thus, we obtain that $\mathcal{M} \models \mathcal{Q}\bar{x}[\dots, \varphi_i(\bar{x}) \vee \psi, \dots]$.

□

We can make a few remarks on this lemma.

Firstly, it states that polarity is a sufficient condition for quantifier shifts equivalences, but is it a necessary condition? As before, the answer is yes if we restrict ourselves to definable subsets, but not in general. This is easy to show: for example, suppose that the equivalence $\mathcal{Q}\bar{x}[\dots, \varphi_i(\bar{x}) \vee \psi, \dots] \equiv \mathcal{Q}\bar{x}[\dots, \varphi_i(\bar{x}), \dots] \vee \psi$ holds for some quantifier Q . Then by applying this to $\psi = \top$, we get $\mathcal{Q}\bar{x}[\dots, \top, \dots] \equiv \top$, which means that $(A_1, \dots, M^{k_i}, \dots, A_n) \in Q(M)$ for all *definable* A_1, \dots, A_n , but it tells us nothing about non-definable subsets, thus Q is definably positive, but possibly not positive in general. This is however not a problem: we only need the sufficient condition part for the study of prenex normal forms presented in the next section.

The second remark we can make is that in the proof of Lemma 3.4, we only used the polarity of Q for one of the two implications. We need full equivalence for prenex normal forms, but it is nonetheless interesting to note that this means we do have the following property for *all* generalized quantifiers, regardless of polarity.

Proposition 3.5 *Let Q be a quantifier, \bar{x} variables, φ_i a formula with free variables \bar{x} , and ψ a formula which does not contain any of the variables \bar{x} . Then:*

- $\mathcal{Q}\bar{x}[\dots, \varphi_i(\bar{x}) \vee \psi, \dots] \models \mathcal{Q}\bar{x}[\dots, \varphi_i(\bar{x}), \dots] \vee \psi$
- $\mathcal{Q}\bar{x}[\dots, \varphi_i(\bar{x}), \dots] \wedge \psi \models \mathcal{Q}\bar{x}[\dots, \varphi_i(\bar{x}) \wedge \psi, \dots]$

Lemma 3.4 provides us with the equivalences we need to push the connectors \vee, \wedge inside quantifiers. We now only have to handle the case of negation. The equivalence

is quite immediate by the properties of the dual quantifiers : we can simply push the negation in/out of a quantifier on argument i by taking the i -th dual Q^{d_i} .

Lemma 3.6 *Let Q be a quantifier, \bar{x} variables, φ_i a formula with free variables \bar{x} . Then:*

$$Q\bar{x}[\dots, \neg\varphi_i(\bar{x}), \dots] \equiv \neg Q^{d_i}\bar{x}[\dots, \varphi_i(\bar{x}), \dots]$$

Proof By definition of the dual quantifier:

$$Q\bar{x}[\dots, \neg\varphi_i(\bar{x}), \dots] \equiv \neg(Q^{d_i}\neg_i)\bar{x}[\dots, \neg\varphi_i(\bar{x}), \dots] \equiv \neg Q^{d_i}\bar{x}[\dots, \varphi_i(\bar{x}), \dots]$$

□

In the case where Q is \forall or \exists , since they are both positive and anti-negative, we find ourselves with the familiar quantifier identities for \forall , \wedge , and the duality $\forall\neg \equiv \neg\exists$.

4 Prenex Normal Forms

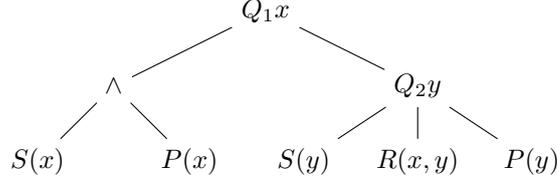
The main objective of this work is to ask whether it is possible to find prenex normal forms for formulas with generalized quantifiers. This is precisely the motivation behind the introduction of quantifier polarities, as they give us quantifiers shift rules which we can use to move quantifiers over the other logical connectors, towards the start of formulas. With the lemmas from the previous section, we now have the results we need to state a prenex normal form theorem.

Before we can do this, we first have to give a proper definition of what is a prenex normal form in the presence of generalized quantifiers. The notion of prenex normal form with generalized quantifiers is a direct generalization of the usual prenex normal form in classical first-order logic, with the main difference being that arbitrary quantifiers are n -ary operators, and thus the notion of a linear quantifier prefix becomes a prefix subtree in the formula tree.

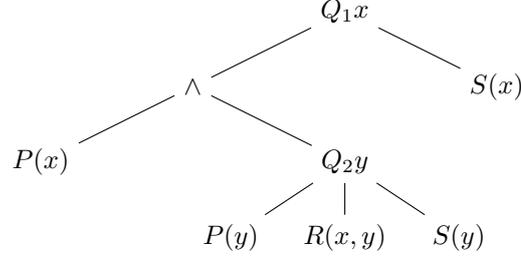
Definition 4.1 *The set of formulas in prenex normal form is defined inductively:*

1. Every atomic formula, i.e. every formula $P(t_1, \dots, t_n)$ where P is a predicate and t_1, \dots, t_n are terms, is in prenex normal form.
2. If φ and ψ are formulas containing no quantifier, then $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \rightarrow \psi$ and $\neg\varphi$ are in prenex normal form.
3. If $\varphi_1, \dots, \varphi_n$ are formulas in prenex normal form and Q is a type $\langle k_1, \dots, k_n \rangle$ quantifier, then $Q\bar{x}[\varphi_1, \dots, \varphi_n]$ is a formula in prenex normal form.
4. No other formula is in prenex normal form.

As an example, the formula $Q_1x[S(x) \wedge P(x), Q_2x[S(y), R(x, y), P(y)]]$ is in prenex normal form:



while the formula $Q_1x[P(x) \wedge Q_2y[P(y), R(x, y), Q(y)], S(x)]$ is not:



Definition 4.2 We say that a quantifier Q is polarized if both of the following hold:

1. Q is either i -positive or i -negative for some argument i .
2. Q is either j -anti-positive or j -anti-negative for some argument j .

Note here that i and j do not necessarily need to be the same argument of Q .

Proposition 4.3 If Q is polarized and i is such that Q has one of the i -polarities of Definition 4.2, then Q^{d_i} is polarized.

Proof Let Q be a polarized quantifier, and let i, j be some arguments of Q satisfying the conditions of Definition 4.2 (possibly not the only ones, and possibly $i = j$).

We have the following cases, which we treat using the square of Lemma 3.3:

- Q is i -positive and j -anti-negative. Then Q^{d_i} is i -anti-negative and j -negative.
- Q is i -positive and j -anti-positive. Then Q^{d_i} is i -anti-negative and j -positive.
- Q is i -negative and j -anti-negative. Then Q^{d_i} is i -anti-positive and j -negative.
- Q is i -negative and j -anti-positive. Then Q^{d_i} is i -anti-positive and j -positive.

In all cases, Q^{d_i} meets the condition for polarization. The same holds if we consider the dual on j , Q^{d_j} instead. \square

Note that this property does not necessarily hold if we take the dual on an arbitrary argument that is neither i nor j .

Essentially, polarization of a quantifier Q ensures us that wherever Q appears in a formula, we are always in one of the cases of Lemma 3.4 so that we can push the

parent connector of \mathcal{Q} inside \mathcal{Q} to prenexify that subformula. Thus for formulas whose quantifiers are polarized, we can state the following theorem.

Theorem 4.4 *Let \mathcal{L} be a first-order language with generalized quantifiers, closed by quantifier duality, and let Φ be a \mathcal{L} -formula such that all quantifiers appearing in Φ are polarized.*

Then there exists a \mathcal{L} -formula Ψ equivalent to Φ such that Ψ is in prenex normal form. Additionally, the only quantifiers appearing in Ψ are quantifiers of Φ and/or duals of quantifiers of Φ .

Proof We prove this by induction on the formulas, using the equivalences from Lemmas 3.4 and 3.6 whenever we encounter a quantifier that needs to be shifted upwards in the formula tree.

The proof itself is straightforward and unsurprising, but it is somewhat lengthy, so we omit the details for now and refer the reader to Appendix A for the full, detailed proof of Theorem 4.4. \square

Two remarks : firstly, the condition on \mathcal{L} to be closed under quantifier duality is important, since if duals are not included in the language, the case of a quantifier within the scope of a negation cannot be treated¹. Secondly, if the language \mathcal{L} contains only polarized quantifiers, the theorem then means that every \mathcal{L} -formula has a prenex normal form.

As in first-order logic with the usual quantifiers, there is of course no unicity of prenex normal forms for a given formula: in general, for a quantifier \mathcal{Q} , if there are multiple arguments i so that \mathcal{Q} satisfies has the adequate i -polarity, we can always choose which of the subformulas of \mathcal{Q} we push the connectors into.

Another remark is that for formulas whose quantifiers are not polarized (or only some of them are), there may or may not exist prenex normal forms depending on the particular quantifier/formula. An example is given by the quantifier $At_Least(M) = \{(A, B) \in \mathcal{P}(M) \times \mathcal{P}(M) \mid |A| \geq |B|\}$: At_Least is positive in the first argument, negative in the second, but neither anti-positive nor anti-negative in any argument. As a result, the formula $At_Least_x[\varphi_1(x), \varphi_2(x)] \vee \psi$ has a prenex normal form²: $At_Least_x[\varphi_1(x) \vee \psi, \varphi_2(x)]$. However, $At_Least_x[\varphi_1(x), \varphi_2(x)] \wedge \psi$ does not have a prenex normal form in general.

5 Characterization of Some Quantifier Classes

Polarities gives us sufficient criteria for prenex normal forms, but many quantifiers do not have these properties. In this section, we present a few results on polarities for some well-known classes of quantifiers.

¹Of course if there is no negation in the language, or even in that specific formula, duals are not really needed, but in the general case we have to include them.

²several, actually: $At_Least_x[\varphi_1(x), \varphi_2(x) \wedge \neg\psi]$ is also an equivalent prenex formula.

5.1 Monotone Quantifiers

Monotonicity, as introduced by [Barwise and Cooper \(1981\)](#), is an important property of generalized quantifiers and has been extensively studied from a linguistic perspective as it is closely linked to the behaviour of quantification in natural language.

Definition 5.1 A type $\langle k_1, \dots, k_n \rangle$ quantifier Q is said to be monotone increasing in the i -th argument if for all sets M , the following holds:

If $(A_1, \dots, A_i, \dots, A_n) \in Q(M)$ and $A_i \subseteq B_i \subseteq M^{k_i}$, then $(A_1, \dots, B_i, \dots, A_n) \in Q(M)$

Conversely, Q is said to be monotone decreasing in i if the following holds:

If $(A_1, \dots, A_i, \dots, A_n) \in Q(M)$ and $B_i \subseteq A_i \subseteq M^{k_i}$, then $(A_1, \dots, B_i, \dots, A_n) \in Q(M)$

In other words, monotone quantifiers are those quantifiers that preserve set inclusion. This is a very common property among natural language quantifiers: it expresses that the quantified sentence remains true when talking about extensions or restrictions of a concept. For example, *All* is 1-decreasing and 2-increasing:

- *If all cats are felines, then all black cats are felines.*
- *If all cats are felines, then all cats are mammals.*

The universe M and the empty set \emptyset being the extreme cases of extension/restriction of a concept, it is not surprising that polarity can be linked to monotonicity, in the following way.

Proposition 5.2 If Q is monotone increasing in the i -th argument, and if for all M and all A_j with $j \neq i$ there exists A_i such that $(A_1, \dots, A_i, \dots, A_n) \in Q(M)$, then Q is i -positive and i -anti-negative.

Proof For any M , if $(A_1, \dots, A_i, \dots, A_n) \in Q(M)$, then since $A_i \subseteq M^{k_i}$, by i -monotonicity of Q we get $(A_1, \dots, M^{k_i}, \dots, A_n) \in Q(M)$. Thus, if for any $(A_j)_{j \neq i}$ with $j \neq i$ there exists such an A_i , we have that for all $(A_j)_{j \neq i}$, $(A_1, \dots, M^{k_i}, \dots, A_n) \in Q(M)$, which by definition means that Q is positive in M . \square

Proposition 5.3 If Q is monotone decreasing in the i -th argument, and if for all M and all A_j with $j \neq i$ there exists A_i such that $(A_1, \dots, A_i, \dots, A_n) \in Q(M)$, then Q is negative and anti-positive in the i -th argument.

Proof This is the symmetrical case to the previous proposition: if Q is decreasing, for all $(A_j)_{j \neq i}$, the existence of A_i such that $(A_1, \dots, M^{k_i}, \dots, A_n) \in Q(M)$ entails that $(A_1, \dots, \emptyset, \dots, A_n) \in Q(M)$ since $\emptyset \subseteq A_i$. \square

The additional condition on the existence of such A_i for polarity is quite restrictive in the general case of n -ary quantifiers: we need to require that $Q(M)$ accepts at least one A_i , for *any* possible choice of the $(A_j)_{j \neq i}$.

However, in the case of unary type $\langle k \rangle$ quantifier, this condition collapses to a much simpler non-triviality condition, only requiring that Q is neither empty, nor is the quantifier that accepts every relation.

Corollary 5.4 *If Q is a type $\langle k \rangle$ quantifier and Q is not trivial, i.e. Q is neither \emptyset nor $\mathcal{P}(M^k)$, then:*

- *If Q is monotone increasing, then Q is positive and anti-negative.*
- *If Q is monotone decreasing, then Q is anti-positive and negative.*

In any case, the additional conditions of non-triviality are important: monotonicity alone is not a sufficient condition for polarity. Consider for example the quantifier "an infinity": $Q^\omega(M) = \{A \subseteq M \mid A \text{ is infinite}\}$. Q^ω is increasing, but is trivial (empty) in all finite sets M , and thus not positive in any finite set. It is, however, anti-negative since \emptyset is never infinite, and it is not trivial if M is infinite, meaning it is positive in all infinite sets.

We can thus conclude that if $\mathcal{L}(Q_1, Q_2, \dots)$ is a language with each Q_i monotone and non-trivial (in the aforementioned sense) in at least one argument, then $\mathcal{L}(Q_1, Q_2, \dots)$ enjoys the prenex normal form theorem 4.4.

5.2 Henkin (Branching) Quantifiers

Henkin quantifiers, also called branching quantifiers were introduced by [Henkin \(1961\)](#) as a way to express independencies between quantified variables beyond the restrictions of first order logic. The underlying idea is to allow a partial ordering of the quantifiers \forall and \exists instead of a total one. The simplest non-trivial example is the quantifier:

$$Q_H = \left(\begin{array}{c} \forall x \exists y \\ \forall u \exists v \end{array} \right)$$

expressing that y depends only on x , and v on u . Formally, they are equivalent to quantifier prefixes of second-order existential logic, as proven by [Enderton \(1970\)](#) and [Walkoe \(1970\)](#): the specific dependencies between variable can be expressed by Skolem functions, for example $\left(\begin{array}{c} \forall x \exists y \\ \forall u \exists v \end{array} \right) F(x, y, u, v)$ is equivalent to $\exists f \exists g \forall x \forall u F(x, f(x), u, g(u))$ ³.

While they are not originally related to the concept of generalized quantifiers, Henkin quantifiers can in fact be seen as a special case of generalized quantifiers. Indeed, a Henkin quantifier Q_H with k variables can be represented by a generalized quantifier of type $\langle k \rangle$, essentially by defining the quantifier with the second-order existential formula that is equivalent to Q_H :

$$Q = \{R \subseteq D^k \mid \exists f_1, \dots, \exists f_m, \forall x_1, \dots, \forall x_n, (x_1, \dots, x_n, f_1(\dots), \dots, f_m(\dots)) \in R\}$$

³Using this specific formulation with functions obviously requires the Axiom of Choice. Note that the AC is however not required to have the equivalence with existential second-order logic: it is also correct to use generic two-place predicates instead, and require that they are functional (which can be expressed by a first-order formula). The use of functions is however often preferred as it is much more intuitive and compact when representing dependencies.

with x_i being the i universally quantified variables, f_j representing the j existentially quantified variables, and the arguments of each f_j being the x_i on which the j -th existential variable depends.

Henkin quantifiers are a very natural extension of usual first-order quantifier prefixes that have applications in proof theory, in particular with regards to Skolemization (see Baaz and Lolić (2021)), and in formal linguistics as studied in Hintikka (1973).

It is fairly easy to see that Henkin quantifiers are monotone and polarized. This should not be a surprise, as Henkin quantifiers are built upon the usual \forall, \exists , and thus are fairly regular as far as generalized quantifiers can be. This was first shown by Kołodziejczyk (2002).

Proposition 5.5 (Kołodziejczyk) *Henkin quantifiers are monotone increasing, positive and anti-negative.*

One can simply remark that Henkin quantifiers are increasing and non-trivial, and thus are polarized by consequence of Corollary 5.4, but it is also quite immediate from the second-order formulation of Henkin quantifiers.

Proof Let Q_H be a Henkin quantifier, with $Q_H(M) = \{R \subseteq M^k \mid \exists f_1, \dots, \exists f_m, \forall x_1, \dots, \forall x_n, (x_1, \dots, x_m, f_1(\dots), \dots, f_m(\dots)) \in R\}$.

Now, $M^k \in Q_H(M)$ iff there are f_1, \dots, f_m functions such that for all $x_1, \dots, x_m, (x_1, \dots, x_m, f_1(\dots), \dots, f_m(\dots)) \in M^k$. M^k containing all possible k -tuples, any choice of functions will do, thus for any set M , $M^k \in Q_H(M)$.

Similarly, there are no f_1, \dots, f_m such that $(x_1, \dots, x_m, f_1(\dots), \dots, f_m(\dots)) \in \emptyset$, thus for all sets M , $\emptyset \notin Q_H(M)$.

By definition, this means Q_H is positive and anti-negative. \square

As a corollary, this means that the (unique) dual Q_H^d of a Henkin quantifier is also monotone increasing, positive and anti-negative, but Q_H^d is *not* a Henkin quantifier: the dual of the prefix $\exists f_1, \dots, \exists f_m, \forall x_1, \dots, \forall x_n$ is $\forall f_1, \dots, \forall f_m, \exists x_1, \dots, \exists x_n$, which does *not* correspond to a Henkin quantifier.

However, if we consider a language including Henkin quantifiers as well as their duals, then all those quantifiers are polarized, and thus enjoy the prenex normal form theorem.

Proposition 5.6 *Let $\mathcal{L}_{\mathcal{H}}$ be a first-order language extended with any number of Henkin quantifiers and closed by duality. Then every $\mathcal{L}_{\mathcal{H}}$ -formula has an equivalent prenex normal form.*

Proof Application of Theorem 4.4. \square

This is not exactly a groundbreaking result: $\mathcal{L}_{\mathcal{H}}$ is equivalent to a fragment of existential second-order logic, which already has a prenex normal form theorem. But the advantage of a direct prenexification in $\mathcal{L}_{\mathcal{H}}$ is that the resulting prenex formula is guaranteed to contain only quantifiers from $\mathcal{L}_{\mathcal{H}}$, which might not be the case

with prenexification in existential second-order logic⁴. To give an example, consider $\mathcal{L}_{\mathcal{H}}$ containing only the simplest Henkin quantifier $\mathcal{Q}_{\mathcal{H}} = (\forall u \exists v \exists y)$, and consider the formula:

$$\Phi = ((\forall x \exists y) \varphi) \wedge ((\forall x' \exists y') \psi)$$

One possible prenexification of Φ using the equivalent second-order formulation is: $\exists f \exists g \exists f' \exists g' \forall x \forall u \forall x' \forall u' (\varphi \wedge \psi)$, where notably the quantifier prefix does *not* correspond to any quantifier prefix in $\mathcal{L}_{\mathcal{H}}$ ⁵.

Meanwhile direct prenexification in $\mathcal{L}_{\mathcal{H}}$ gives either one of the prefixes $(\forall x \exists y)(\forall x' \exists y')$ or $(\forall x' \exists y')(\forall x \exists y)$ which are both in $\mathcal{L}_{\mathcal{H}}$.

6 Conclusion

In this work, we studied the conditions for the existence of prenex normal forms in first-order languages extended with generalized quantifiers. To do so, we introduced the notions of *i*-polarity of a quantifier, and showed that polarities give us sufficient conditions for syntactic properties of quantifiers, allowing one to shift quantifiers outward and connectors inward from subformulas. As a consequence, we established prenex normal form theorems for any extended first-order language whose quantifiers have the adequate polarity properties.

While a significant portion of quantifiers *in general* do not particularly have any polarity properties, we nonetheless established that some of the more well-known and common classes of quantifiers enjoy these properties and thus have prenex normal forms theorems. In particular, this includes monotone quantifiers, which are a very significant class of quantifiers, especially in natural language quantification, as well as Henkin quantifiers.

In the continuation of this line of study, we aim to examine more in detail some other common classes of quantifiers that could fall under the scope of the results presented here. From a broader perspective, prenex normal forms theorems are a useful result in usual first-order logic for the study of proof systems and automated theorem proving. It would be interesting to see if this work could enable such approaches for generalized quantifiers, and if the various syntactic properties we established could help in designing reasonable proof systems, a topic which is significantly underexplored in the broader case of logics with generalized quantifiers.

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⁴As it can be more expressive than $\mathcal{L}_{\mathcal{H}}$.

⁵It corresponds to a Henkin quantifier, but in a strictly more expressive language than $\mathcal{L}_{\mathcal{H}}$.

Appendix A Proof of the Prenex Normal Form Theorem

We give here the full details of the proof of Theorem 4.4.

Theorem A.1 *Let \mathcal{L} be a first-order language with generalized quantifiers, closed by quantifier duality, and let Φ be a \mathcal{L} -formula such that all quantifiers appearing in Φ are polarized.*

Then there exists a \mathcal{L} -formula Ψ equivalent to Φ such that Ψ is in prenex normal form. Additionally, the only quantifiers appearing in Ψ are quantifiers of Φ and/or duals of quantifiers of Φ .

Definition A.2 *We define the complexity $c(\Phi)$ of a formula Φ as its number of quantifiers and connectors, excluding \neg and \neg_i :*

- If Φ is an atomic formula, then $c(\Phi) = 0$.
- If $\Phi = \neg\varphi$, $c(\Phi) = c(\varphi)$.
- If $\Phi = \varphi \star \psi$, with $\star \in \{\vee, \wedge, \rightarrow\}$, then $c(\Phi) = 1 + c(\varphi) + c(\psi)$.
- If $\Phi = \mathcal{Q}\bar{x}[\varphi_1, \dots, \varphi_n]$, then $c(\Phi) = 1 + \sum_{i=1}^n c(\varphi_i)$.

We need to omit negations from the complexity measure, since the equivalences of Lemma 3.4 can add or remove negations in some cases.

Proof of Theorem 4.4 We proceed by induction on the *complexity* of the formula Φ .

Every atomic formula and every negation of an atomic formula is already in prenex normal form, thus every formula of complexity 0 is itself a prenex formula of complexity 0.

Let n be a strictly positive integer, and suppose that for all $k < n$, every formula of complexity k whose quantifiers are polarized has an equivalent formula of complexity at most k in prenex normal form, whose quantifiers are also all polarized.

Let Φ be a formula of complexity n whose quantifiers are polarized. Then Φ is in one of the following forms:

1. $\Phi = \mathcal{Q}\bar{x}[\varphi_1, \dots, \varphi_m]$.

If $c(\Phi) = n$, then $\sum_{j=1}^m c(\varphi_j) = n - 1$, meaning every subformula φ_j has complexity lower than n .

We can thus apply our induction hypothesis to every φ_j to obtain ψ_1, \dots, ψ_m formulas in prenex normal form, with complexity at most equal to the corresponding φ_j , such that Φ is equivalent to $\Psi = \mathcal{Q}\bar{x}[\psi_1, \dots, \psi_m]$. By definition, Ψ is in prenex normal form, and since for all j , $c(\psi_j) \leq c(\varphi_j)$, its complexity is at most n . Additionally, by hypothesis all quantifiers in each ψ_j are polarized, so this is also true of Ψ .

2. $\Phi = \neg\mathcal{Q}\bar{x}[\varphi_1, \dots, \varphi_m]$.

Similarly, if $c(\Phi) = n$, then $\sum_{j=1}^m c(\varphi_j) = n - 1$ since the leading negation does not add to the complexity, and there are ψ_1, \dots, ψ_m formulas in prenex normal form such that Φ is equivalent to $\neg\mathcal{Q}\bar{x}[\psi_1, \dots, \psi_m]$. \mathcal{Q} is either positive or negative in some

i , so we push the negation inwards on the i -th argument: by Lemma 3.6 Φ is equivalent to $\mathcal{Q}^{d_i}\bar{x}[\psi_1, \dots, \neg\psi_i, \dots, \psi_m]$. $c(\neg\psi_i) \leq n - 1$, thus we can transform $\neg\psi_i$ into ψ'_i in prenex normal form, such that Φ is equivalent to $\Psi = \mathcal{Q}^{d_i}\bar{x}[\psi_1, \dots, \psi'_i, \dots, \psi_m]$.

By definition, Ψ is in prenex normal form (all the quantifier subformulas are in prenex normal form), and its complexity is at most $c(\Phi) = n$. Since we chose to push the negation inwards into the i -th argument and \mathcal{Q} has an i -polarity, Property 4.3 ensures that \mathcal{Q}^{d_i} is also polarized, and so are all quantifiers in Ψ .

3. $\Phi = \varphi \vee \psi$.

φ, ψ have complexity less than n , since $c(\varphi) + c(\psi) = n - 1$, and thus by hypothesis there are equivalent formulas φ', ψ' in prenex normal form such that $c(\varphi') + c(\psi') \leq n - 1$.

If neither φ' nor ψ' starts with a quantifier, then by definition of the prenex normal form, they contain no quantifiers. Then $\Psi = \varphi' \vee \psi'$ is in prenex normal form, equivalent to Φ and its complexity is at most n . In this case, Ψ also does not contain any quantifier.

Suppose instead that one of φ' or ψ' starts with a quantifier. Without loss of generality, suppose φ' is in the form $\mathcal{Q}\bar{x}[\varphi_1, \dots, \varphi_m]$, with \mathcal{Q} polarized.

If \mathcal{Q} is positive in some argument i , then by Lemma 3.4, $\mathcal{Q}\bar{x}[\varphi_1, \dots, \varphi_m] \vee \psi'$ is equivalent to $\mathcal{Q}\bar{x}[\dots, \varphi_i \vee \psi', \dots]$, provided we rename the variables bound by \mathcal{Q} so that there is no conflict with variables in ψ . We have $\sum_{j=1}^m c(\varphi_j) \leq c(\varphi') - 1$, thus:

$$c(\varphi_i \vee \psi') + \sum_{j \neq i} c(\varphi_j) \leq c(\varphi') + c(\psi') \leq n - 1$$

If instead \mathcal{Q} is negative in some argument i , then by Lemma 3.4, $\mathcal{Q}\bar{x}[\varphi_1, \dots, \varphi_m] \vee \psi'$ is equivalent to $\mathcal{Q}\bar{x}[\dots, \varphi_i \wedge \neg\psi', \dots]$, again renaming bound variables if needed. The complexity of the resulting formula is the same as above, since $c(\varphi_i \wedge \neg\psi') = c(\varphi_i \vee \psi')$

In either case, we can use our induction hypothesis on every φ_j , $\varphi_i \vee \psi'$ and $\varphi_i \wedge \neg\psi'$, exactly as in cases 1. and 2.. This means that there are formulas in prenex normal form ψ_1, \dots, ψ_m such that $\varphi' \vee \psi'$ is equivalent to $\Psi = \mathcal{Q}\bar{x}[\psi_1, \dots, \psi_m]$, and $\sum_{j=1}^m c(\psi_j) \leq n - 1$. Additionally, by hypothesis all quantifiers in ψ_1, \dots, ψ_m are also polarized. Thus, Ψ is equivalent to Φ , is of complexity at most n , and since every ψ_j is in prenex normal form, Ψ is also in prenex normal form.

4. $\Phi = \neg(\varphi \vee \psi)$.

In this case, Φ is equivalent to $\neg\varphi \wedge \neg\psi$. Since $c(\neg\varphi) = c(\varphi) < n$ and $c(\neg\psi) = c(\psi) < n$, we can apply the induction hypothesis: there are φ', ψ' in prenex normal form such that Φ is equivalent to $\varphi' \wedge \psi'$.

If neither φ' nor ψ' starts with a quantifier, then $\varphi' \wedge \psi'$ is already in prenex normal form, and its complexity is at most n . Else, if either φ' or ψ' starts with a quantifier \mathcal{Q} , using the same reasoning as before, we obtain that there are ψ_1, \dots, ψ_m formulas in prenex normal form, such that Φ is equivalent to $\Psi = \mathcal{Q}\bar{x}[\psi_1, \dots, \psi_m]$, with $\sum_{j=1}^m c(\psi_j) \leq n - 1$.

5. $\Phi = \varphi \wedge \psi$ or $\Phi = \neg(\varphi \wedge \psi)$.

These cases are treated in the exact same way as $\varphi \vee \psi$ and $\neg(\varphi \vee \psi)$, using the other equivalences from Lemma 3.4.

6. $\Phi = \varphi \rightarrow \psi$ is treated as $\neg\varphi \vee \psi$, and is covered by the other cases.

By induction, we obtain that for all n , every formula Φ of complexity n whose quantifiers are polarized has an equivalent formula Ψ in prenex normal form of complexity at most n . Since in the process of transforming Φ into Ψ , for every quantifier in Φ , we either leave it unchanged or transform it into one of its dual Q^{d_i} in a way that Q^{d_i} is also polarized, it results that the only quantifiers appearing in Ψ are those of Φ or duals of quantifiers in Φ , and that they are also all polarized. \square

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