Generalized Quantifiers and Prenex Normal Forms

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Abstract. Generalized quantifiers, introduced by Mostowski and Lindström, are generalizations of the standard quantifiers of modern logic, ∀ and ∃, and have been extensively studied for their applications to model theory and computational linguistics, but not so much for their purely syntactical and proof-theoretic properties. In this paper, we introduce the notion of polarity of a quantifier, and use it as criteria for the existence of prenex normal forms for some classes of quantifiers.

Keywords: Logic · Generalized Quantifiers · Prenex Normal Forms

1 Introduction

Generalized quantifiers, initially introduced by Mostowski [7] and further developed by Lindström [6] are a generalization of standard first-order quantification that treats quantifiers as generic relations over relations (or tuples of relations) on the domain (a.k.a universe, model) of an interpretation: thus, quantifiers are viewed as higher-order concepts.

This concept of generalized quantifiers is not new, and has been extensively studied from a model-theoretic and linguistic viewpoint. This is not surprising, as it originates in the generalization of the usual quantifiers' interpretation in a given model, to generic relations on this model. The expressive power of this formalism has also found a significant use in linguistics, for its ability to account for the great diversity of quantification in natural language. However, very little work has been done on the more proof-theoretic and syntactic aspects of generalized quantifiers. Their expressive power is part of the reason why: finding sensible syntactic properties and reasonable deduction systems for non-trivial quantifier classes *in general* is considerably more challenging. Among the few works on proof theory for some very limited subsets of quantifiers, we can cite Keisler [5] on quantifiers such as "uncountably many", and more recently the works of Baaz and Lolic [1] on Henkin quantifiers.

In a very broad sense, the general aim of our work is to continue these proof-theoretical approaches of quantifiers, and if it is most likely impossible for the full theory of generalized quantifiers, to find classes of quantifiers for which we can have reasonable properties and deduction systems. In this article specifically, we study the possible existence of prenex normal forms for languages with general quantifiers. The concept of prenex normal forms (i.e. putting quantifiers at the start of the formula), is an important syntactic property for logical languages and is widely used in automated theorem proving, which makes it an interesting question if we want to develop such systems for generalized quantifiers.

2 A Reminder on Generalized Quantifiers

We will start by giving here a very succinct presentation of generalized quantifiers, mainly inspired by Peters and Westerståhl [8], and Szymanik [9]. We refer the reader to these works for a more thorough description.

Definition 1 (Syntax). *If $k_1, ..., k_n$ are strictly positive integers, a* ***quantifier*** *Q* ***of type*** *$\langle k_1, ..., k_n \rangle$ is a variable-binding operator that applies to n formulas, binding k_i variables in the i-th formula. Thus, if $\bar{x}_1, ..., \bar{x}_n$ are tuples of variables and $\varphi_1, ..., \varphi_n$ are formulas, then*

$$Q\bar{x}_1...\bar{x}_n[\varphi_1(\bar{x}_1), ..., \varphi_n(\bar{x}_n)]$$

is a formula, in which all free occurrences of $\bar{x}_i$ in φ_i are bound by Q.

Thus, given a language $\mathcal{L}$, we can extend it into a language $\mathcal{L}'$ by adding a number of generalized quantifiers following the aforementioned rule. In the remainder of this work, we will place ourselves within the framework of a standard first-order language augmented with generalized quantifiers.[1]

Definition 2 (Semantics). *Semantically, a quantifier Q of type $\langle k_1, ..., k_n \rangle$ is interpreted by a function which maps each model $\mathcal{M} = (D, I)$ to a n-ary relation $Q^{\mathcal{M}}$ over n k_i-ary relations on D (in other words, $Q^{\mathcal{M}}$ is a set of subsets of $D^{k_1} \times ... \times D^{k_n}$).*

The satisfaction relation for Q in the model $\mathcal{M} = (D, I)$ is then as follows:

$$\mathcal{M} \models Q\bar{x}_1...\bar{x}_n[\varphi_1(\bar{x}_1), ..., \varphi_n(\bar{x}_n)] \quad \textit{iff} \quad (\varphi_1^{\mathcal{M}}, ..., \varphi_n^{\mathcal{M}}) \in Q^{\mathcal{M}}$$

where $\varphi_i^{\mathcal{M}} = \{\bar{x} \in D^{k_i} \mid \mathcal{M} \models \varphi_i(\bar{x})\}$

The relation $Q^{\mathcal{M}}$ for each model is sometimes called a *local* quantifier, and the function Q mapping each model $\mathcal{M}$ to $Q^{\mathcal{M}}$ is in turn called a *global* quantifier. In cases where it does not lead to any ambiguity, we can usually conflate the two and speak of the quantifier Q with no distinction.

In the same manner, when there is no ambiguity in which variables are bound by Q, we can omit them and write in a more concise, set-oriented manner: $Q[\varphi_1, ..., \varphi_n]$ to mean $Q\bar{x}_1...\bar{x}_n[\varphi_1(\bar{x}_1), ..., \varphi_n(\bar{x}_n)]$

This formal definition is better understood with some examples. In simple terms, the idea is that predicates are seen as sets of individuals (or tuples of individuals), and quantifiers as relations over those sets. Thus, to say that $\exists x F(x)$ is

[1] There is of course a sensible notion of generalized quantifiers on other languages, for example second-order or higher-order languages, but this is well beyond the scope of this study.

to say that the set of x that have the property F is not empty, meaning that the quantifier $\exists$ is in every model $\mathcal{M} = (D, I)$ the relation $\exists^{\mathcal{M}} = \{A \subseteq D \mid A \neq \emptyset\}$. Similarly, the universal quantifier is simply the relation that only accepts the whole domain D: $\forall^{\mathcal{M}} = \{D\}$. Building on this idea, any relation on n predicates of respective arities $k_1, ..., k_n$ defines a quantifier of type $\langle k_1, ..., k_n \rangle$. For example: $Some^{\mathcal{M}} = \{A, B \subseteq D \times D \mid A \cap B \neq \emptyset\}$ of type $\langle 1, 1 \rangle$, meaning that some individuals with property A also have the property B.

Definition 3. *If Q and Q' are two quantifiers of type $\langle k_1, ...k_n \rangle$, then $Q \vee Q'$ and $Q \wedge Q'$ are quantifiers of type $\langle k_1, ...k_n \rangle$, with the expected meaning:*

- $\mathcal{M} \models (Q \vee Q')[\varphi_1, ..., \varphi_n]$ *iff* $\mathcal{M} \models Q[\varphi_1, ..., \varphi_n]$ *or* $\mathcal{M} \models Q'[\varphi_1, ..., \varphi_n]$
- $\mathcal{M} \models (Q \wedge Q')[\varphi_1, ..., \varphi_n]$ *iff* $\mathcal{M} \models Q[\varphi_1, ..., \varphi_n]$ *and* $\mathcal{M} \models Q'[\varphi_1, ..., \varphi_n]$

Definition 4. *If Q is a quantifier of type $\langle k_1, ...k_n \rangle$, then the* ***outer negation*** *of Q is the quantifier $\neg Q$ such that:*

$$\mathcal{M} \models (\neg Q)[\varphi_1, ..., \varphi_n] \quad \text{iff} \quad \mathcal{M} \not\models Q[\varphi_1, ..., \varphi_n]$$

The ***inner negation*** *on the i-th argument is the quantifier $Q\neg_i$ defined by:*

$$\mathcal{M} \models (Q\neg_i)[\varphi_1, ..., \varphi_i, ..., \varphi_n] \quad \text{iff} \quad \mathcal{M} \models Q[\varphi_1, ..., \neg\varphi_i, ..., \varphi_n]$$

From this, the notion of dual quantifier can naturally be extended to any quantifier, by defining the dual of Q in the i-th argument as $Q^{d_i} = (\neg Q)\neg_i = \neg(Q\neg_i)$.[2]

3 Introducing i-polarities

In order to study prenex normal forms for generalized quantifiers, we need to introduce the notion of *polarity* of a quantifier in a given argument i.

Definition 5. *Let Q be a quantifier of type $\langle k_1, ..., k_n \rangle$. We say that Q is:*

- ***positive*** *in the i-th argument if for all $\mathcal{M}$ and for all φ_k with $k \neq i$:*

$$\mathcal{M} \models Q[..., \varphi_{i-1}, \top, \varphi_{i+1}, ...]$$

- ***anti-positive*** *in the i-th argument if for all $\mathcal{M}$ and for all φ_k with $k \neq i$:*

$$\mathcal{M} \not\models Q[..., \varphi_{i-1}, \top, \varphi_{i+1}, ...]$$

- ***negative*** *in the i-th argument if for all $\mathcal{M}$ and for all φ_k with $k \neq i$:*

$$\mathcal{M} \models Q[..., \varphi_{i-1}, \bot, \varphi_{i+1}, ...]$$

- ***anti-negative*** *in the i-th argument if for all $\mathcal{M}$ and for all φ_k with $k \neq i$:*

$$\mathcal{M} \not\models Q[..., \varphi_{i-1}, \bot, \varphi_{i+1}, ...]$$

[2] One can easily verify that for all i, $(Q^{d_i})^{d_i} = Q$, but in general for $i \neq j$, $(Q^{d_i})^{d_j} \neq Q$.

An important remark to make here is that the definition of polarity is given with respect to *definable* sets, thus dependent on the particular choice of a language $\mathcal{L}$: in the definition, the φ_k range over all *extensions* of $\mathcal{L}$-formulas, i.e. $\mathcal{L}$-definable subsets of $\mathcal{M}$. This means that the notion of polarity is dependent on the language, and in different languages an quantifier may not have the same polarity. This is not a problem: in a given language we still have a sensible notion of polarity with the properties that follow, but it is important to be aware of this dependency on the choice of a language.

A first immediate property of polarity is that Q is positive (or negative) iff $\neg Q$ is anti-positive (or anti-negative), and that Q is positive (or anti-positive) iff $Q\neg_i$ is negative (or anti-negative).[3]

It is also worth noting that while a quantifier cannot be positive *and* anti-positive (nor negative *and* anti-negative) in the same argument, it may well happen that it is both positive and negative (or anti-positive and anti-negative) and in general, polarity on one argument is independent of polarity on other arguments. For quantifiers that have these polarity properties, we can establish quantifier shift rules, as follows.

Lemma 1. *Let Q be a quantifier, $\bar{x}$ variables, φ_i a formula with free variables $\bar{x}$, and F a formula which does not contain any of the variables $\bar{x}$. We have the equivalences:*

1. *If Q is positive in the i-th argument, then*

$$Q\bar{x}[..., \varphi_i(\bar{x}) \vee F, ...] \equiv Q\bar{x}[..., \varphi_i(\bar{x}), ...] \vee F$$

2. *If Q is anti-positive in the i-th argument, then*

$$Q\bar{x}[..., \varphi_i(\bar{x}) \vee F, ...] \equiv Q\bar{x}[..., \varphi_i(\bar{x}), ...] \wedge \neg F$$

3. *If Q is negative in the i-th argument, then*

$$Q\bar{x}[..., \varphi_i(\bar{x}) \wedge F, ...] \equiv Q\bar{x}[..., \varphi_i(\bar{x}), ...] \vee \neg F$$

4. *If Q is anti-negative in the i-th argument, then*

$$Q\bar{x}[..., \varphi_i(\bar{x}) \wedge F, ...] \equiv Q\bar{x}[..., \varphi_i(\bar{x}), ...] \wedge F$$

Proof. We will only present here the proof for the first case (Q positive). The proof for the other cases is unremarkably similar, or can be obtained by simply using the properties on the positivity/negativity of $\neg Q$ and $Q\neg_i$.

Let Q be a quantifier, positive in the i-th argument.

$\Rightarrow$: Let $\mathcal{M} = (D, I)$ be a model, and suppose that $\mathcal{M} \models Q\bar{x}[..., \varphi_i(\bar{x}) \vee F, ...]$. If $\mathcal{M} \models F$, then trivially $\mathcal{M} \models Q\bar{x}[..., \varphi_i(\bar{x}), ...] \vee F$.

[3] The important corollary is that Q is i-positive iff Q^{d_i} is i-anti-negative, and so on.

If $\mathcal{M} \not\models F$, then $\{\bar{x} \in D^{k_i} \mid \mathcal{M} \models \varphi_i(\bar{x}) \vee F\} = \{\bar{x} \in D^{k_i} \mid \mathcal{M} \models \varphi_i(\bar{x})\}$. Thus, from the hypothesis $\mathcal{M} \models Q\bar{x}[..., \varphi_i(\bar{x}) \vee F, ...]$, we have also $\mathcal{M} \models Q\bar{x}[..., \varphi_i(\bar{x}), ...]$, which means $\mathcal{M} \models Q\bar{x}[..., \varphi_i(\bar{x}), ...] \vee F$.

$\Leftarrow$: Let $\mathcal{M} = (D, I)$ be a model, and suppose that $\mathcal{M} \models Q\bar{x}[..., \varphi_i(\bar{x}), ...] \vee F$.

If $\mathcal{M} \models F$, then $\{\bar{x} \in D^{k_i} \mid \mathcal{M} \models \varphi_i(\bar{x}) \vee F\} = \{\bar{x} \in D^{k_i} \mid \mathcal{M} \models \top\}$. Since Q is taken to be positive, we obtain that $\mathcal{M} \models Q\bar{x}[..., \varphi_i(\bar{x}) \vee F, ...]$.

If $\mathcal{M} \not\models F$, then by hypothesis, necessarily $\mathcal{M} \models Q\bar{x}[..., \varphi_i(\bar{x}), ...]$. Since $\mathcal{M} \not\models F$, we also have $\{\bar{x} \in D^{k_i} \mid \mathcal{M} \models \varphi_i(\bar{x}) \vee F\} = \{\bar{x} \in D^{k_i} \mid \mathcal{M} \models \varphi_i(\bar{x})\}$. Thus, we obtain that $\mathcal{M} \models Q\bar{x}[..., \varphi_i(\bar{x}) \vee F, ...]$. □

Lemma 2. *Let Q be a quantifier, $\bar{x}$ variables, φ_i a formula with free variables $\bar{x}$. Then:*

$$Q\bar{x}[..., \neg\varphi_i(\bar{x}), ...] \equiv \neg Q^{d_i}\bar{x}[..., \varphi_i(\bar{x}), ...]$$

Proof. By definition of the dual quantifier:
$Q\bar{x}[..., \neg\varphi_i(\bar{x}), ...] \equiv \neg Q^{d_i}\neg_i\bar{x}[..., \neg\varphi_i(\bar{x}), ...] \equiv \neg Q^{d_i}\bar{x}[..., \varphi_i(\bar{x}), ...]$ □

4 Prenex Normal Forms

The next logical step is to ask whether it is possible, using these rules, to find prenex normal forms for formulas with generalized quantifiers. The notion of prenex normal form with generalized quantifiers is a direct generalization of the usual prenex normal form in classical first-order logic, with the main difference being that arbitrary quantifiers are n-ary operators, and thus the notion of a linear quantifier prefix becomes a prefix subtree in the formula tree.

Definition 6. *The set of formulas in* ***prenex normal form*** *is defined inductively:*

1. *Every atomic formula, i.e. every formula $P(t_1, ..., t_n)$ where P is a predicate and $t_1, ..., t_n$ are terms, is in prenex normal form.*
2. *If φ and ψ are formulas containing no quantifier, then $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \rightarrow \psi$ and $\neg\varphi$ are in prenex normal form.*
3. *If $\varphi_1, ..., \varphi_n$ are formulas in prenex normal form and Q is a type $\langle k_1, ..., k_n \rangle$ quantifier, then $Q[\varphi_1, ..., \varphi_n]$ is a formula in prenex normal form.*
4. *No other formula is in prenex normal form.*

Theorem 1. *Let $\mathcal{L}$ be a first-order language with generalized quantifiers and closed by quantifier duality, that is if Q is a n-ary quantifier of $\mathcal{L}$ then for $1 \leq i \leq n$, $Q^{d_i} \in L$, and let Φ be a $\mathcal{L}$-formula verifying the following property (H):*

(H) *For all quantifiers Q in Φ, both of the following holds:*

1. *Q is either positive or negative in i for some argument i.*
2. *Q is either anti-positive or anti-negative in j for some argument j.*

Then there exists a $\mathcal{L}$-formula Ψ equivalent to Φ such that Ψ is in prenex normal form.

Two remarks : firstly, the condition on $\mathcal{L}$ to be closed under quantifier duality is important, since if duals are not included in the language, the case of a quantifier within the scope of a negation can not be treated. Secondly, we could alternatively require that all quantifiers in $\mathcal{L}$ satisfy (H), in which case all $\mathcal{L}$-formulas will have a prenex normal form (note that if Q satisfies (H), then so do all the duals of Q).

Proof. The proof is by induction on the **complexity** of the formula Φ, which we define by the number of quantifiers and logical connectors excluding $\neg$, $\neg_i$ in Φ, noted $c(\Phi)$.

Every atomic formula and every negation of an atomic formula is already in prenex normal form, thus every formula of complexity 0 has an equivalent prenex formula of complexity 0.

Let n be a strictly positive integer, and suppose that for all $k < n$, every formula of complexity k that satisfies (H) has an equivalent formula of complexity at most k in prenex normal form. Let Φ be a formula satisfying (H) of complexity n. Then Φ is in one of the following forms:

- $\Phi = F \vee G$.
  F, G verify $c(F)+c(G) = n-1$, and thus there are equivalent formulas F', G' in prenex normal form such that $c(F') + c(G') \leq n-1$.
  If neither F' nor G' starts with a quantifier, then by definition of the prenex normal form, they contain no quantifiers. Then $\Psi = F' \vee G'$ is in prenex normal form, equivalent to Φ and its complexity is at most n. Suppose now that one of F' or G' starts with a quantifier. Without loss of genericity, suppose F' is in the form $Q[\varphi_1, ..., \varphi_m]$.
  If Q is positive in some argument i, then by Lemma 1, $F' \vee G'$ is equivalent to $Q[..., \varphi_i \vee G', ...]$. We have $\sum_{j=1}^{m} c(\varphi_j) \leq c(F') - 1$, thus:
  $$c(\varphi_i \vee G') + \sum_{j \neq i} c(\varphi_j) \leq c(F') + c(G') \leq n-1$$
  If instead Q is negative in some argument i, then by Lemma 1, $F' \vee G'$ is equivalent to $Q[..., \varphi_i \wedge \neg G', ...]$, provided we rename the variables bound by Q so that there is no conflict with variables in G. The complexity of the resulting formula is the same as above, since $c(\varphi_i \wedge \neg G') = c(\varphi_i \vee G')$
  In either case, we can use our induction hypothesis on every φ_j, $\varphi_i \vee G'$ and $\varphi_i \wedge \neg G'$. This means that there are formulas in prenex normal form $\psi_1, ..., \psi_m$ such that $F' \vee G'$ is equivalent to $\Psi = Q[\psi_1, ..., \psi_m]$, and $\sum_{j=1}^{m} c(\psi_j) \leq n-1$
  Thus, Ψ is equivalent to Φ, is of complexity at most n, and since every ψ_j is in prenex normal form, Ψ is also in prenex normal form.
- $\Phi = \neg(F \vee G)$.
  The method used in the previous case is also applicable here, since the negation does not increase the complexity of Φ. Thus, there are F', G' in prenex normal form such that Φ is equivalent to $\neg(F' \vee G')$.

If neither F' nor G' starts with a quantifier, then $\neg(F' \vee G')$ is already in prenex normal form, and its complexity is at most n. Else, using the same proof as before, we obtain that there are $\psi_1, ..., \psi_m$ formulas in prenex normal form, such that Φ is equivalent to $\Psi = \neg Q[\psi_1, ..., \psi_m]$, with $\sum_{j=1}^{m} c(\psi_j) \leq n-1$.
It remains only to push the negation inwards. Q is either positive or negative in some argument i, and $\neg Q[\psi_1, ..., \psi_m]$ is equivalent to $Q^{d_i}[\psi_1, ..., \neg\psi_i, ..., \psi_m]$. If Q is a quantifier satisfying hypothesis (H) on argument i, then so is Q^{d_i}. Since $c(\neg\psi_i) = c(\psi_i) \leq n-1$, we can transform $\neg\psi_i$ into an equivalent formula in prenex normal form ψ_i' of complexity at most $c(\neg\psi_i)$. Thus, Φ is equivalent to $\Psi = Q^{d_i}[\psi_1, ..., \psi_i', ..., \psi_m]$. All immediate subformulas of Ψ are in prenex normal form, so Ψ is in prenex normal form, and $c(\Psi) \leq n$.

– $\Phi = F \wedge G$ or $\Phi = \neg(F \wedge G)$.
  These cases are treated in the exact same way as $F \vee G$ and $\neg(F \vee G)$, using the other equivalences from Lemma 1.

– $\Phi = Q[\varphi_1, ..., \varphi_m]$.
  If $c(\Phi) = n$, then: $\sum_{j=1}^{m} c(\varphi_j) = n-1$.
  We can then apply our induction hypothesis to every φ_j to obtain $\psi_1, ..., \psi_m$ formulas in prenex normal form such that Φ is equivalent to $\Psi = Q[\psi_1, ..., \psi_m]$. By definition, Ψ is in prenex normal form, and since for all j, $c(\psi_j) \leq c(\varphi_j)$, its complexity is at most n.

– $\Phi = \neg Q[\varphi_1, ..., \varphi_m]$.
  Similarly, if $c(\Phi) = n$, then: $\sum_{j=1}^{m} c(\varphi_j) = n-1$, and there are $\psi_1, ..., \psi_m$ formulas in prenex normal form such that Φ is equivalent to $\neg Q[\psi_1, ..., \psi_m]$. Q is either positive or negative in some i, so we push the negation inwards on the i-th argument: Φ is equivalent to $Q^{d_i}[\psi_1, ..., \neg\psi_i, ..., \psi_m]$. $c(\neg\psi_i) \leq n-1$, thus we can transform $\neg\psi_i$ into ψ_i' in prenex normal form, such that Φ is equivalent to $\Psi = Q^{d_i}[\psi_1, ..., \psi_i', ..., \psi_m]$.
  By definition, Ψ is in prenex normal form (all the quantifier subformulas are in prenex normal form), and its complexity is at most $c(\Phi) = n$.

By induction, we obtain that for all n, every formula of complexity n satisfying (H) has an equivalent formula in prenex normal form of complexity at most n. □

Let us make some remarks on this result. Firstly, there is of course no unicity of prenex normal forms for a given formula (as usual): in general, for a quantifier Q, if there are multiple arguments i so that Q satisfies (H), we can always "choose" which of the subformulas of Q we push the connectors into.

Another remark is that for formulas that do not satisfy (H) (or only partly), there may or may not exist prenex normal forms depending on the case. An example is given by the quantifier $At\_Least[A, B]$ meaning $|A| \geq |B|$: At_Least is positive in the first argument, negative in the second, but neither anti-positive nor anti-negative in any argument. As a result, the formula $At\_Least[\varphi_1, \varphi_2] \vee F$

has a prenex normal form[4]: $At\_Least[\varphi_1 \vee F, \varphi_2]$. However, $At\_Least[\varphi_1, \varphi_2] \wedge F$ does not in general (note that in particular cases, depending on the formula F and/or the language considered, there may still be an equivalent prenex formula).

5 Characterization of Some Quantifier Classes

Polarities gives us strong criteria for prenex normal forms, but many quantifiers do not have these properties. In this section, we present a few results on polarities for some well-known classes of quantifiers.

5.1 Monotone Quantifiers

Monotonicity, as introduced by Barwise and Cooper [2], is an important property of generalized quantifiers and has been extensively studied from a linguistic perspective as it is closely linked to the behaviour of quantification in natural language.

Definition 7. *A quantifier Q of type $\langle k_1, ..., k_n \rangle$ is said to be* ***monotone increasing*** *in the i-th argument if for all $\mathcal{M} = (D, I)$, the following holds:*

If $\mathcal{M} \models Q[\varphi_1, ..., \varphi_n]$ *and* $\varphi_i^{\mathcal{M}} \subseteq \varphi_i'^{\mathcal{M}} \subseteq D^{k_i}$, *then* $\mathcal{M} \models Q[\varphi_1, ..., \varphi_i', ..., \varphi_n]$

with $\varphi_i^{\mathcal{M}} == \{\bar{x} \in D^{k_i} \mid \mathcal{M} \models \varphi_i(\bar{x})\}$ *denoting the extension of* φ_i *in* $\mathcal{M}$.[5]

Conversely, Q is said to be ***monotone decreasing*** *in i if the following holds:*

If $\mathcal{M} \models Q[\varphi_1, ..., \varphi_n]$ *and* $\varphi_i'^{\mathcal{M}} \subseteq \varphi_i^{\mathcal{M}} \subseteq D^{k_i}$, *then* $\mathcal{M} \models Q[\varphi_1, ..., \varphi_i', ..., \varphi_n]$

Now, since for a given extension of a formula $\varphi_i^{\mathcal{M}}$, we have $\bot^{\mathcal{M}} \subseteq \varphi_i^{\mathcal{M}} \subseteq \top^{\mathcal{M}}$, from the definitions of monotonicity and polarity we can make the following remarks.

Property 1 *If Q is monotone increasing in the i-th argument, and if for all $\mathcal{M}$ and all φ_j with $j \neq i$ there exists φ_i such that $\mathcal{M} \models Q[\varphi_1, ..., \varphi_i, ..., \varphi_n]$, then Q is positive and anti-negative in the i-th argument.*

Property 2 *If Q is monotone decreasing in the i-th argument, and if for all $\mathcal{M}$ and all φ_j with $j \neq i$ there exists φ_i such that $\mathcal{M} \models Q[\varphi_1, ..., \varphi_i, ..., \varphi_n]$, then Q is negative and anti-positive in the i-th argument.*

Corollary 1. *If Q is a quantifier of type $\langle k \rangle$ and Q is not trivial (i.e. Q is neither $\emptyset$ nor $\mathcal{P}(D^k)$), then:*

- *If Q is monotone increasing, then Q is positive and anti-negative.*
- *If Q is monotone decreasing, then Q is anti-positive and negative.*

[4] several, actually: $At\_Least[\varphi_1, \varphi_2 \wedge \neg F]$ is also an equivalent prenex formula.

[5] In other terms, $\varphi_i^{\mathcal{M}} \subseteq \varphi_i'^{\mathcal{M}}$ means that $\mathcal{M} \models \varphi_i \rightarrow \varphi_i'$

5.2 Henkin (Branching) Quantifiers

Henkin (or branching) quantifiers were introduced by Henkin [4] as a way to express independencies between quantified variables beyond the restrictions of first order logic. The underlying idea is to allow a partial ordering of the quantifiers $\forall$ and $\exists$ instead of a total one. The simplest non-trivial example is the quantifier $Q_H = \left(\begin{smallmatrix}\forall x & \exists y \\ \forall u & \exists v\end{smallmatrix}\right)$, meaning that y depends only on x, and v on u. Formally, they are equivalent to quantifier prefixes of second-order existential logic, as proven by Enderton [3] and Walkoe [10]: the specific dependencies between variable can be expressed by Skolem functions, for example $\left(\begin{smallmatrix}\forall x & \exists y \\ \forall u & \exists v\end{smallmatrix}\right) F(x, y, u, v)$ is equivalent to $\exists f \exists g \forall x \forall u F(x, f(x), u, g(u))^6$.

While they are not originally related to the concept of generalized quantifiers, Henkin quantifiers can in fact be seen as a special case of generalized quantifiers. Indeed, a Henkin quantifier with k variables is represented by a generalized quantifier of type $\langle k \rangle$:

$$Q = \{R \subseteq D^k | \; \exists f_1, ..., \exists f_m, \forall x_1, ..., \forall x_n, (x_1, ..., x_n, f_1(...), ..., f_m(...)) \in R\}$$

with x_i being the i universally quantified variables, f_j representing the j existentially quantified variables, and the arguments of each f_j being the x_i on which the j-th existential variable depends.

Theorem 2. *We consider a first-order language $\mathcal{L}$ augmented with Henkin quantifiers (not necessarily all of them) and closed by quantifier duality.*

Then Henkin quantifiers in $\mathcal{L}$ are positive and anti-negative, and thus every $\mathcal{L}$-formula has a prenex normal form.

Proof. For any Henkin quantifier, let Q_H be its representation as a generalized quantifier: $Q_H = \{R \subseteq D^k | \; \exists f_1, ..., \exists f_m, \forall x_1, ..., \forall x_n, (x_1, ..., x_m, f_1(...), ..., f_m(...)) \in R\}$.

Now, $\mathcal{M} \models Q_H[\top]$ if there are $f_1, ..., f_n$ functions such that for all $x_1, ..., x_m$, $(x_1, ..., x_m, f_1(...), ..., f_m(...)) \in D^k$. D^k containing all possibles tuples, any choice of functions will do, thus for all models $\mathcal{M}$, $\mathcal{M} \models Q_H[\top]$.

Similarly, there are no $f_1, ..., f_n$ such that $(x_1, ..., x_m, f_1(...), ..., f_m(...)) \in \emptyset$, thus for all models $\mathcal{M}$, $\mathcal{M} \not\models Q_H[\bot]$.

By definition, this means Q_H is positive and anti-negative, and Theorem 1 gives us the existence of prenex normal forms. $\square$

6 Conclusion

We have introduced the notion of i-polarity of generalized quantifiers, which gives us criteria for the existence of prenex normal forms, and we have begun

[6] Using this specific formulation with functions obviously requires the Axiom of Choice. Note that the AC is however not required to have the equivalence with existential second-order logic: it is also correct to use generic two-place predicates instead. The use of functions is however often preferred as it is much more intuitive when representing dependencies.

studying well-known classes of quantifiers in the light of this new concept. In the future, we aim to pursue this characterization of some common quantifiers classes. We have also noticed that these criteria are sufficient, but not necessary: many other quantifier formulas can be put in prenex normal form, and identifying other conditions for this to be possible is the natural continuation of this work.

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